Derivation of Various Dual Formulations in Magnetostatics via Error Based Energy Approach

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Abstract – **Various mixed or conventional dual formulations in magnetostatics can be derived via minimization of the error functional of the constitutive law written in energy terms and different ways of considering Maxwell's equations. This demonstration allows a better understanding of the equivalence and the differences between these formulations. The computational effort and performance of different formulations are described.**

Index terms – **Magnetostatics, finite element modeling, dual formulations, Whitney elements.**

I. Introduction

The interest in dual formulations in electromagnetic field computation is numerous. In static fields, dual formulations provide complementary energy bounds which permit calculation of global quantities such as impedances with minimum computational cost. Regarding the accuracy of results, there exists also a complementary feature: one formulation gives good results of the magnetic flux or the electromotive forces while the other provides better results for magnetomotive forces or currents. In addition, the dual formulations provide an efficient tool for the estimation of the local computation errors. This is helpful for adaptive mesh refinement.

Different dual formulations in terms of potential or field variables have been developed. The most commonly used formulations are the conventional scalar potential and the vector potential formulations. Mixed dual formulations have drawn much attention of some authors in recent years. They work directly with the field variables whereas the potentials are introduced in weak forms [1]. The dual mixed systems can be solved either simultaneously [2] or separately [3].

In this paper, we intend to show that all these formulations can be derived from the minimisation of the error function of constitutive laws via the energy approach. Another type of dual formulations based on the use of the tree technique to ensure Maxwell's equations, can also be established. These formulations, having advantage of reduced number of unknowns and no need of gauge condition, do not however attract much attention in the community.

The present work allows a better understanding of the theoretical and numerical equivalence as well as differences of these diverse formulations. The computation effort and the performance of these formulations can be compared.

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II. Minimisation of Error Functional

Consider a magnetostatic problem in a bounded region Ω. The boundary of Ω is split in two: $\partial\Omega = \Gamma_b \cup \Gamma_h$ and the intersection of Γ_b and Γ_h is empty. For simplicity of description, we assume the following boundary conditions hold : $\mathbf{n}\cdot\mathbf{b}=0$ on Γ_b and $\mathbf{n}\times\mathbf{h}=0$ on Γ_h. This problem can be solved by minimising the following functional:

$$\int_\Omega \varepsilon_w(\mathbf{h},\mathbf{b})\,d\Omega = \int_\Omega \{\int_0^{\mathbf{b}} \mathbf{h}\cdot d\mathbf{b} + \int_0^{\mathbf{h}} \mathbf{b}\cdot d\mathbf{h} - \mathbf{b}\cdot\mathbf{h}\}d\Omega = \min, \quad (1)$$

where ε_w is an error function of the constitutive law [4] as illustrated in the cross-hatched region in Fig.1.

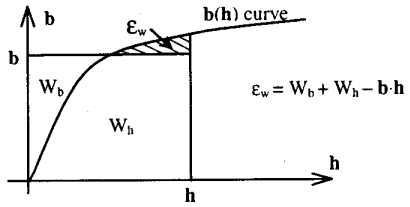

Fig.1. Error function of the constitutive law.

Taking the variation of (1) with respect to **b** and **h**, respectively, leads to two coupled equations:

$$\int_\Omega \frac{1}{\mu}\mathbf{b}\cdot\delta\mathbf{b}\,d\Omega - \int_\Omega \mathbf{h}\cdot\delta\mathbf{b}\,d\Omega = 0 \quad (2.a)$$

$$-\int_\Omega \mathbf{b}\cdot\delta\mathbf{h}\,d\Omega + \int_\Omega \mu\mathbf{h}\cdot\delta\mathbf{h}\,d\Omega = 0 \quad (2.b)$$

Considering the continuity of the fields, **b** and **h** should be solved in the div-conform and curl-conform spaces, respectively,

$$S_{div} = \{\mathbf{b} \in IL^2(\Omega),\ \mathrm{div}\,\mathbf{b} \in L^2(\Omega)\},$$

$$S_{curl} = \{\mathbf{h} \in IL^2(\Omega),\ \mathrm{curl}\,\mathbf{h} \in IL^2(\Omega)\},$$

where L^2 and IL^2 are the Hilbert spaces of square integrable scalar and vector fields over Ω, respectively.

Equations (2) must hold for any variation of **b** and **h**. Using the Galerkin approach, the variations $\delta\mathbf{b}$ and $\delta\mathbf{h}$ can be replaced, respectively, by the test functions $\mathbf{b}' \in S_{div}$ and $\mathbf{h}' \in S_{curl}$. The spaces S_{div} and S_{curl} are to be approximated by appropriate finite elements.

There exists a set of natural adapted elements, named Whitney elements, to interpolate different electromagnetic variables [5]. They are the first order nodal elements W^0,

which ensure C^0 continuity of a scalar variable; the edge elements W^1 and the facet elements W^2 which match, respectively, the tangential continuity and the normal continuity of a vector variable; and the piecewise constant volume elements W^3. To approximate S_{div} and S_{curl}, the Whitney face and edge elements W^2 and W^1 are two suitable spaces.

Equations (2) must be solved together with Maxwell's equations in magnetostatics: div $\mathbf{b} = 0$, curl $\mathbf{h} = \mathbf{j}$ and the boundary conditions. Depending on how the Maxwell equations are matched, we obtain different formulations.

Before the derivation of formulations, we express first the excitation current $\mathbf{j}$ by a source field (current vector potential) $\mathbf{t}$ such that curl $\mathbf{t} = \mathbf{j}$. It can be noted that $\mathbf{t}$ defined in this way is not unique, but any field fulfilling curl $\mathbf{t} = \mathbf{j}$ works. A particular case is calculating $\mathbf{t}$ using Biot-Savart's law. In this case, $\mathbf{t}$ is calculated in the whole region and this can be time consuming. The most convenient way is to set $\mathbf{t}$ in a simply connected region Ω_t including the excitation coil with the boundary condition $\mathbf{n} \times \mathbf{t} = 0$ on $\partial\Omega_t$ and to calculate it using a finite element approximation [6]. After doing that, the magnetic field $\mathbf{h}$ is split into a curl free field $\mathbf{h}_r$ and the source field $\mathbf{t}$: $\mathbf{h} = \mathbf{h}_r + \mathbf{t}$. Equations (2) become

$$\int_\Omega \frac{1}{\mu}\mathbf{b}'\cdot\mathbf{b}\,d\Omega - \int_\Omega \mathbf{b}'\cdot\mathbf{h}_r\,d\Omega = \int_{\Omega_t} \mathbf{b}'\cdot\mathbf{t}\,d\Omega \qquad (2'.a)$$

$$-\int_\Omega \mathbf{h}'\cdot\mathbf{b}\,d\Omega + \int_\Omega \mu\mathbf{h}'\cdot\mathbf{h}_r\,d\Omega = -\int_{\Omega_t} \mu\mathbf{h}'\cdot\mathbf{t}\,d\Omega \qquad (2'.b)$$

III. Mixed Formulation with Simultaneous Solution

Maxwell's equations can be imposed as constraints using Lagrange multipliers. In this case, (2'.a) and (2'.b) are solved simultaneously together with the constraint equations [2]:

Find $\mathbf{b} \in W_b^2$, $\mathbf{h}_r \in W_h^1$, $\mathbf{a} \in W_h^2$ and $\phi \in W^3$ such that

$$\int_\Omega \frac{1}{\mu}\mathbf{b}'\cdot\mathbf{b}\,d\Omega - \int_\Omega \mathbf{b}'\cdot\mathbf{h}_r\,d\Omega + \int_\Omega \operatorname{div}\mathbf{b}'\phi\,d\Omega = \int_{\Omega_t} \mathbf{b}'\cdot\mathbf{t}\,d\Omega \qquad \forall\,\mathbf{b}' \in W_b^2 \qquad (3.a)$$

$$-\int_\Omega \mathbf{h}'\cdot\mathbf{b}\,d\Omega + \int_\Omega \mu\mathbf{h}'\cdot\mathbf{h}_r\,d\Omega + \int_\Omega \operatorname{curl}\mathbf{h}'\cdot\mathbf{a}\,d\Omega = -\int_{\Omega_t} \mu\mathbf{h}'\cdot\mathbf{t}\,d\Omega \qquad \forall\,\mathbf{h}' \in W_h^1 \qquad (3.b)$$

$$\int_\Omega \phi'\operatorname{div}\mathbf{b}\,d\Omega = 0 \qquad \forall\,\phi' \in W^3 \qquad (3.c)$$

$$\int_\Omega \mathbf{a}'\cdot\operatorname{curl}\mathbf{h}_r\,d\Omega = 0 \qquad \forall\,\mathbf{a}' \in W_h^2 \qquad (3.d)$$

The subscript b on W^2, h on W^1 and on W^2 means the boundary conditions are included in Whitney element spaces:

$$W_b^2 = \{\mathbf{b} \in W^2 \mid \mathbf{n}\cdot\mathbf{b} = 0 \text{ on } \Gamma_b\}$$

$$W_h^1 = \{\mathbf{h} \in W^1 \mid \mathbf{n}\times\mathbf{h} = 0 \text{ on } \Gamma_h\}$$

$$W_h^2 = \{\mathbf{a} \in W^2 \mid \mathbf{n}\cdot\mathbf{a} = 0 \text{ on } \Gamma_h\}$$

The Lagrange multipliers $\mathbf{a}$ and ϕ in (3) are nothing else but the magnetic vector and scalar potentials. Their approximation is not conformal because $\mathbf{a}$ is approximated in W^2 instead of W^1 and ϕ is in W^3 (volume elements) instead of W^0 (nodal elements). Therefore the name mixed formulation.

This formulation was reported in [2]. But the multipliers given there are $\mathbf{a}/\mu$ and $\mu\phi$ instead of $\mathbf{a}$ and ϕ. In our opinion, this introduction of the permeability μ in Lagrange multipliers is not necessary.

The magnetic vector potential $\mathbf{a}$ given in this formulation is not unique. A gauge condition is necessary to ensure its uniqueness. Two solutions are possible. The first is to impose the gauge condition div $\mathbf{a} = 0$ as a constraint using again the Lagrange multiplier [2]. An additional multiplier λ is then introduced. Equation (3.d) is modified to (3'.d), and the equation (3.e) is added.

$$\int_\Omega \mathbf{a}'\cdot\operatorname{curl}\mathbf{h}_r\,d\Omega + \int_\Omega \operatorname{div}\mathbf{a}'\lambda\,d\Omega = 0 \qquad \forall\,\mathbf{a}' \in W_h^2 \qquad (3'.d)$$

$$\int_\Omega \lambda'\cdot\operatorname{div}\mathbf{a}\,d\Omega = 0 \qquad \forall\,\lambda' \in W^3 \qquad (3.e)$$

Another possibility is to include the gauge condition div $\mathbf{a} = 0$ in the facet element space W^2. This can be done by reducing the facet element space to a set of independent facets [3]; or by further expressing $\mathbf{a}$ by the curl of another potential. The reduction of facet element space can be achieved using a tree technique as described in section V.

It must be noted that the gauge condition for $\mathbf{a}$ is not indispensable from the numerical point of view. The system still converges without the gauge condition [7].

The whole matrix system resulting from this formulation is symmetrical and semi-positive definite. It is a very big system. The number of unknowns is $n_f^b + n_e^h + n_f^h + 2n_t$ if (3'.d) and (3.e) are used for the uniqueness of $\mathbf{a}$. It is $n_f^b + n_e^h + n_f^h$ if the facet elements space is reduced to ensure the gauge condition, or $n_f^b + n_e^h + n_f^h + n_t$ if no gauge condition is imposed. n_f^b and n_f^h are numbers of facets excluding those on the boundary Γ_b and Γ_h, n_e^h is the number of edges except those on Γ_h and n_t the number of tetrahedra.

IV. Dual Mixed Formulations

Let us express $\mathbf{h}_r = -\operatorname{grad}\phi$ in (2'.a), and solve div $\mathbf{b} = 0$ over Ω. We get the following mixed formulation:

Find $\mathbf{b} \in W_b^2$ and $\phi \in W^3$ such that

$$\int_\Omega \frac{1}{\mu}\mathbf{b}'\cdot\mathbf{b}\,d\Omega - \int_\Omega \operatorname{div}\mathbf{b}'\phi\,d\Omega = \int_{\Omega_t} \mathbf{b}'\cdot\mathbf{t}\,d\Omega \qquad \forall\,\mathbf{b}' \in W_b^2 \qquad (4.a)$$

$$\int_\Omega \phi'\operatorname{div}\mathbf{b}\,d\Omega = 0 \qquad \forall\,\phi' \in W^3 \qquad (4.b)$$

This formulation ensures the normal continuity of $\mathbf{b}$. The approximation of ϕ in W^3 is not conformal. The flux conservation law div $\mathbf{b} = 0$ is strongly imposed in an average sense over the domain Ω.

Similarly, writing $\mathbf{b}$ = curl $\mathbf{a}$ in (2'.b), and solving curl $\mathbf{h}_r$ = 0, leads to a mixed formulation, dual of the preceding one:

Find $\mathbf{h}_r \in W_h^1$ and $\mathbf{a} \in W_h^2$ such that

$$\int_\Omega \mu \mathbf{h}' \cdot \mathbf{h}_r \, d\Omega - \int_\Omega \text{curl } \mathbf{h}' \cdot \mathbf{a} \, d\Omega = -\int_{\Omega_t} \mu \mathbf{h}' \cdot \mathbf{t} \, d\Omega \qquad \forall \mathbf{h}' \in W_h^1 \quad (5.a)$$

$$\int_\Omega \mathbf{a}' \cdot \text{curl } \mathbf{h}_r \, d\Omega = 0 \qquad \forall \mathbf{a}' \in W_h^2 \quad (5.b)$$

The approximation of $\mathbf{h}_r$ in W^1 is conformal and ensures the tangential continuity of $\mathbf{h}$. The curl free condition of $\mathbf{h}_r$ is ensured on average over Ω. The approximation of $\mathbf{a}$ is not conformal. A gauge condition such that div $\mathbf{a}$ =0 can be imposed to guarantee the uniqueness of $\mathbf{a}$ as described at the end of the section III.

These mixed formulations are suggested by [1] and developed in [3][7]. In reference [1], they are derived from a bilateral approach using Tonti's diagram. In [3], it is mentioned that the formulation (4.a) comes from the weak form of the equation $\mathbf{h} = \mathbf{h}_r - \text{grad } \phi$, and the formulation (5.a) is the weak form of the equation $\mathbf{b}$ = curl $\mathbf{a}$. A little difference in (5) can also be noted. In our approach, the current density is represented by the source field curl $\mathbf{t}$. Whereas in [1] and [3], the current density is directly included in the formulation:

$$\int_\Omega \mu \mathbf{h}' \cdot \mathbf{h} \, d\Omega - \int_\Omega \text{curl } \mathbf{h}' \cdot \mathbf{a} \, d\Omega = 0 \qquad \forall \mathbf{h}' \in W_h^1 \quad (5'.a)$$

$$\int_\Omega \mathbf{a}' \cdot \text{curl } \mathbf{h} \, d\Omega = \int_\Omega \mathbf{a}' \cdot \mathbf{j} \, d\Omega \qquad \forall \mathbf{a}' \in W_h^2 \quad (5'.b)$$

The two matrix systems of dual mixed formulations (4) and (5) are symmetrical and semi positive definite. The number of degrees of freedom of the formulation (4) is $n_f^b + n_t$ and in (5), it is $n_e^h + n_f^h + n_t$ or $n_e^h + n_f^h - n_t$ depending how the uniqueness of $\mathbf{a}$ is ensured, or $n_e^h + n_f^h$ if the gauge condition is not imposed.

V. Dual Formulations using Tree Techniques

Another way of taking into account div $\mathbf{b}$ = 0 and curl $\mathbf{h}_r$ = 0 is by introducing them, respectively, in the spaces W_b^2 and W_h^1. These spaces are defined as:

$$W_b^2{}_0 = \{\mathbf{b} \in W_b^2 \mid \text{div } \mathbf{b} = 0 \text{ in } \Omega\},$$

$$W_h^2{}_0 = \{\mathbf{h} \in W_h^1 \mid \text{curl } \mathbf{h} = 0 \text{ in } \Omega\}.$$

Noting that $W_b^2{}_0$ is the kernel of div in W_b^2 and $W_h^1{}_0$ is the kernel of curl in W_h^1. They are two orthogonal spaces, i.e.

$$\int_\Omega \mathbf{u} \cdot \mathbf{v} \, d\Omega = 0, \text{ for } \mathbf{u} \in W_b^2{}_0 \text{ and } \mathbf{v} \in W_h^1{}_0$$

Solving $\mathbf{b}$ in $W_b^2{}_0$, and $\mathbf{h}_r$ in $W_h^1{}_0$, the two equations (2'.a) and (2'.b) are uncoupled due to the orthogonalty. We get respectively, a $\mathbf{b}$-oriented formulation and a $\mathbf{h}$-oriented formulation.

Find $\mathbf{b} \in W_b^2{}_0$, such that

$$\int_\Omega \frac{1}{\mu} \mathbf{b}' \cdot \mathbf{b} \, d\Omega = \int_{\Omega_t} \mathbf{b}' \cdot \mathbf{t} \, d\Omega, \qquad \forall \mathbf{b}' \in W_b^2{}_0 \quad (6)$$

Find $\mathbf{h}_r \in W_h^1{}_0$ such that

$$\int_\Omega \mu \mathbf{h}' \cdot \mathbf{h}_r \, d\Omega = -\int_{\Omega_t} \mu \mathbf{h}' \cdot \mathbf{t} \, d\Omega \qquad \forall \mathbf{h}' \in W_h^1{}_0 \quad (7)$$

The spaces $W_b^2{}_0$ and $W_h^1{}_0$ can be defined using the spanning tree technique. Taking first the case of edge element space W_h^1. The space $W_h^1{}_0$ is constructed by a spanning tree, i.e., a set of edges spanning all nodes but forming no loops. The degrees of freedom are associated with the branches of the tree. In a mesh of n_n nodes, the number of branches of the tree is n_n–1. Considering the boundary condition on Γ_h, the number of unknowns is n_n^h. This is the number of nodes excluding those on Γ_h.

The same idea can be applied in W_b^2 to built $W_b^2{}_0$. The degrees of freedom are a set of independent facets that do not form a closed surface. It can be proven that the number of independent facets, in the case of a simply connected domain containing no cavity, is the number of facets minus the number of tetrahedra: $n_f - n_t$. According to Euler's identity, this is equal to $n_e - n_n + 1$, i.e. the number of branches of the co-tree. Removing the degrees of freedom from Γ_b, the number of unknowns is $n_e^b - n_n^b$, n_e^b and n_n^b are the number of edges and nodes except those on the boundary Γ_b.

VI. Dual Formulations in Terms of Potentials

The space $W_b^2{}_0$ can also be approximated by the curl of the edge element space W^1, because curl W^1 is the kernel of the divergence in W^2. Similarly, $W_h^1{}_0$ can be approximated by the gradient of the nodal elements W^0, since grad W^0 is the kernel of the curl in W^1. This approximation amounts to introducing potential variables such that $\mathbf{b}$ = curl $\mathbf{a}$ and $\mathbf{h}_r = -$ grad ϕ. We get the following conventional potential formulations:

Find $\mathbf{a} \in W_b^1$ such that

$$\int_\Omega \frac{1}{\mu} \text{curl} \mathbf{a}' \cdot \text{curl} \mathbf{a} \, d\Omega = \int_{\Omega_t} \text{curl} \mathbf{a}' \cdot \mathbf{t} \, d\Omega \qquad \forall \mathbf{a}' \in W_a^1 \quad (8)$$

Find $\phi \in W_h^0$ such that

$$\int_\Omega \mu \text{grad} \phi' \cdot \text{grad} \phi \, d\Omega = \int_{\Omega_t} \mu \text{grad} \phi' \cdot \mathbf{t} \, d\Omega \qquad \forall \phi' \in W_h^0 \quad (9)$$

The main inconvenience of the formulation in terms of potentials is the necessity of a gauge condition to ensure their uniqueness. The gauge condition in (9) is easy. It is only necessary to fix the value of ϕ on the boundary Γ_h. The number of unknowns is n_n^h, the same as in formulation (7).

The gauge condition in (8) can be set by $\mathbf{a} \cdot \mathbf{w} = 0$, where $\mathbf{w}$ is a field formed by the edges forming a tree [8]. The number of unknowns is equal to the number of branches of the co-tree, except those on Γ_b: $n_e^b - n_n^b$. It is the same as in formulation (6). We note that the vector potential formulation

obtained in such a manner is automatically compatible [6], because the current density is expressed by the curl of a source field and this source field is projected on the curl of edge element space. In this case, the system converges when using an iterative solver without an explicit gauge [6].

VII. Discussion and Conclusions

The minimization of the error energy functional leads to two dual equations in terms of **b** and **h**. Depending on how Maxwell's equations are taken into account, we get different formulations. These formulations can be divided in two categories: the mixed formulations (3) and (4), (5), and the ordinary unmixed formulations (6), (7) and (8), (9).

In the mixed formulations, Maxwell's equations are taken into account either by the constraint equations using Lagrange multipliers; or by expressing **h** and **b** in potential variables and approximating them in weak forms. In these formulations, Maxwell's equations curl **h** = **j** and div **b** = 0 are satisfied, although strongly because no integration by parts is performed to reduce the order of partial derivatives, but in average over the domain. The mixed formulation using Lagrange multiplier method leads to a very big algebraic system with all field and potential variables as unknowns. Although we can obtain **b** and **h** at once, the computation cost for the inversion of the system is very high. The system is split in two ((4) and (5)) if $\mathbf{h}_r$ in (2'.a) and **b** in (2'.b) are expressed by potentials. The formulations (4) and (5) are nearly equivalent to (3), but from the point of view of computation cost, it is much more economical to solve two smaller systems than solving one big system.

The advantage of the mixed formulations is that they work directly with the field variables and avoid some problems encountered in conventional potential formulations such as the cuts in the scalar potential formulation. The main inconvenience is that the number of unknowns is very high. Appropriate solvers must also used to solve non-positive definite matrix system. Otherwise, in the mixed formulations, gauge condition for the uniqueness of the vector potential is a matter to be considered.

In the ordinary unmixed formulations, the equations curl $\mathbf{h}_r$ = 0 and div **b** = 0 are introduced in the Whitney elements spaces. This can be done either by introducing potential variables or by using tree techniques. The introduction of potentials leads to conventional scalar and vector potential formulations (9) and (8). The problem to be solved in the scalar potential formulation is the need of cuts to avoid multi-valued potentials in the case of multiply connected regions. This problem can be eliminated if the source field **t** is set in a simply connected region containing conductors. In the vector potential formulation, the most cumbersome problem is the necessity of gauge to ensure the uniqueness of **a**. It has been proven that explicit gauge is not indispensable when using an iterative solver and if the system is compatible [6]. Because in this condition, the system converges to a solution and any solution of **a** provides a unique solution of **b**. However, in the case of a direct inversion method, a gauge condition is necessary because the matrix is singular without gauges.

Using the tree techniques to account for the divergence free of **b** and curl free of $\mathbf{h}_r$ leads to the dual formulations (6) and (7). This enables working directly with field variables. The number of unknowns is exactly the same as for the corresponding potential formulations but the cumbersome gauge condition for the vector potential is avoided. It should be noted that although the number of unknowns in these formulations is the same as those of the potential formulations, the conditioning of resulting matrices may different. Also, these matrices are denser than those for potential formulations, because the interaction is between the branches of the tree which form a loop (or a closed surface) together with a branch of the co-tree instead of the interaction between elements in the conventional formulations. The sparsity of the matrices depends on how the tree is constructed. The construction of the tree must be optimal in order to minimize the number of non-zero terms in the matrix. These formulations are of practical interest, because no gauge condition is needed and the field quantities are obtained directly. Moreover, these formulations can be easily applied to solve problems in multiply connected domain or in domain containing cavities (case of electrostatic field, for example), provided that additional components are added in the tree to form necessary cuts or links [9][10].

It has been shown in this paper that different dual formulations can be derived from the minimization of the error energy functional. Maxwell's equations are strongly satisfied in different manners in these dual formulations and the errors lie on the constitutive law. Working with dual formulations is of considerable interest as indicated in the introduction. The difference, the equivalence as well as the efficiency of these formulations have been analyzed and compared.

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