Impedance Boundary Conditions for Transient Scattering Problems

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Abstract - **The approximate boundary conditions for the tangential component of the electric field and normal component of the magnetic field on the surface of a homogeneous body of finite conductivity (conductor or lossy dielectric) are derived for transient incident electromagnetic field. Scale factors for basic variables are introduced in such a way, that a small parameter, proportional to the ratio of the penetration depth and body's characteristic size, appears in the dimensionless Maxwell's equations for the conducting region and then the perturbation method is used. The use of the boundary conditions together with space-time domain surface integral equations for electric and magnetic fields is proposed. A numerical example is included to illustrate the theory.**

I. INTRODUCTION

More than fifty years ago Rytov [1] applied the perturbation method to calculate the skin effect in good conductor. He obtained the distributions of magnetic and electric fields inside the conductor perpendicular to the surface in the form of a power series in the skin depth δ

$$\delta = \sqrt{2/(\omega\mu\sigma)} \qquad (1)$$

where ω is the angular frequency, σ the electrical conductivity and μ the magnetic permeability of the conductor. Relation between the first non-zero terms of the expansions of the electric and magnetic fields on the surface of conducting body is well-known as Impedance Boundary Condition:

$$\vec{n}\times\vec{E}(\omega) = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}(\vec{n}\times\vec{H})\times\vec{n} = Z(\omega)\left(\vec{n}\times\vec{H}(\omega)\right)\times\vec{n} \qquad (2)$$

where $\vec{E}$ and $\vec{H}$ are the electric and magnetic fields, respectively; $\vec{n}$ is normal vector pointing inside the conductor; ε is the electrical permeability and Z is the surface impedance. Relation (2) is frequently used in both of low- and high-frequency analysis to represent the conducting region if the penetration depth δ is much smaller than the geometrical dimension D of the conductor region.

This approach can be used for transient problems, when, for instance, the pulse duration τ_p of the incident field is so short that the field has no time to diffuse deep into the body and remains concentrated in a thin layer near the body's surface. In this case the use of the inverse Laplace transform allows (2) to be written in the time domain [2,3]:

$$\vec{n}\times\vec{E}(t) = Z(t) * \left(\vec{n}\times\vec{H}(t)\right)\times\vec{n} \qquad (3)$$

$$Z(t) = \left(\frac{\mu}{\varepsilon}\right)^{1/2}\left\{\delta(t) + \frac{\sigma}{2\varepsilon}\left[I_1\left(\frac{\sigma t}{2\varepsilon}\right) - I_0\left(\frac{\sigma t}{2\varepsilon}\right)\right]\exp\left(-\frac{\sigma t}{2\varepsilon}\right)\right\}$$

where * denotes a time-domain convolution product, $I_n(x)$ is the modified Bessel function of order n and $\delta(t)$ is the unit step function. For a high conducting body, the condition (3) reduced to

$$\vec{n}\times\vec{E}(t) = \left(\frac{2\mu}{\pi\sigma}\right)^{1/2}\int_0^t \vec{n}\times\left(\vec{n}\times\frac{\partial\vec{H}(t-t')}{\partial t}\right)t'^{-1/2}\,dt' \qquad (3a)$$

Recently implementation of (3) in the FDTD and finite element methods has been proposed for transient scattering problems [2]-[5]. Tesche [6] implemented the condition (3a) in the space-time domain electric field surface integral equation. In the present paper, the condition (3) for the tangential component of the electric field is supplemented by the condition for the normal component of the magnetic field on the body's surface and both of them are implemented in the space-time domain magnetic field surface integral equation as well as electric field surface integral equation.

II. TRANSIENT IMPEDANCE BOUNDARY CONDITIONS FOR LOSSY DIELECTRIC BODY

Consider a homogeneous body of finite conductivity surrounded by the non-conductive medium. The magnetic permeability of the body and the exterior medium is constant and equal to μ_0. The electric and magnetic fields inside the body can be described by Maxwell's equations in the following form

$$\nabla\times\vec{H} = \sigma\vec{E} + \varepsilon_R\varepsilon_0\frac{\partial\vec{E}}{\partial t} \quad (4) \qquad \nabla\times\vec{E} = -\mu_0\frac{\partial\vec{H}}{\partial t} \quad (5)$$

where body's relative permeability ε_R is assumed constant.

Let the time variation of the incident field be such that the penetration depth δ into the body remains small as compared with the minimum radius D of the curvature of the surface of the body

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$$\delta = \sqrt{\tau_p/\sigma\mu_0} << D$$

We introduce a local Cartesian coordinate system related to the surface of the body. We direct the coordinates ξ_1, ξ_2 along the surface and the coordinate η normal to it inside the body. The characteristic lengths associated with these coordinates are D and δ, respectively.

Following the theory of the perturbation methods, we now transform to dimensionless variables of order *O(1)* by choosing the following scale factors:

$$[\xi]=D; \quad [\eta]=pD; \quad [H]=ID^{-1}; \quad [E]=\mu_0 I(\tau_p)^{-1};$$
$$p=(\tau_p(\sigma\mu_0)^{-1})^{1/2}D^{-1}; \quad q=D(c\tau_p)^{-1} \tag{6}$$

where c is the velocity of light in vacuum; I is the scale for the current. The quantity p in (6) is a small parameter since it is the ratio of two characteristic times of the problem, namely, the duration of the pulse and the time required for the field to diffuse over the distance D. The parameter q is the electric dimension of the body in transient case.

With the variables (6) Eqs.(4)-(5) are written as follows

$$p^2\frac{\partial H_\eta}{\partial \xi_2} - p\frac{\partial H_{\xi_2}}{\partial \eta} = E_{\xi_1} + a\frac{\partial E_{\xi_1}}{\partial t} \tag{7a}$$

$$p\frac{\partial H_{\xi_1}}{\partial \eta} - p^2\frac{\partial H_\eta}{\partial \xi_1} = E_{\xi_2} + a\frac{\partial E_{\xi_2}}{\partial t} \tag{7b}$$

$$p^2\left(\frac{\partial H_{\xi_2}}{\partial \xi_1} - \frac{\partial H_{\xi_1}}{\partial \xi_2}\right) = E_\eta + a\frac{\partial E_\eta}{\partial t} \tag{7c}$$

$$p\frac{\partial E_\eta}{\partial \xi_2} - \frac{\partial E_{\xi_2}}{\partial \eta} = -p\frac{\partial H_{\xi_1}}{\partial t} \tag{8a}$$

$$\frac{\partial E_{\xi_1}}{\partial \eta} - p\frac{\partial E_\eta}{\partial \xi_1} = -p\frac{\partial H_{\xi_2}}{\partial t} \tag{8b}$$

$$\frac{\partial E_{\xi_2}}{\partial \xi_1} - \frac{\partial E_{\xi_1}}{\partial \xi_2} = -\frac{\partial H_\eta}{\partial t} \tag{8c}$$

where $a = p^2 q^2 \varepsilon_R = \varepsilon_0 \varepsilon_R/(\sigma\tau_p)$

From (7c) it follows that $E_\eta = O(p^2)$. By taking into account this estimation and neglecting the terms of the order $O(p^2)$, we re-write Eqs. (7a)-(7b) and (8a)-(8b) in the following form

$$-p\frac{\partial H_{\xi_2}}{\partial \eta} = E_{\xi_1} + a\frac{\partial E_{\xi_1}}{\partial t} \quad (9a) \qquad \frac{\partial E_{\xi_2}}{\partial \eta} = p\frac{\partial H_{\xi_1}}{\partial t} \quad (10a)$$

$$p\frac{\partial H_{\xi_1}}{\partial \eta} = E_{\xi_2} + a\frac{\partial E_{\xi_2}}{\partial t} \quad (9b) \qquad \frac{\partial E_{\xi_1}}{\partial \eta} = -p\frac{\partial H_{\xi_2}}{\partial t} \quad (10b)$$

The tangential components of the electric field on the body's surface are derived from (10a) and (10b) by integrating over the boundary layer

$$E^0_{\xi_k} = p(-1)^{k+1}\partial F_{3-k}/\partial t \qquad k=1,2 \tag{11}$$

where

$$F_k(\xi_1,\xi_2,t) = \int_0^\infty H_{\xi_k}(\xi_1,\xi_2,\eta,t)d\eta$$

and superscript "0" denotes the values on the body's surface (η=0).

The normal component of the magnetic field can be derived from (8c) as follows

$$\vec{n}\cdot\vec{H}^0 = p\sum_{i=1}^{2}\partial F_i/\partial\xi_i \tag{12}$$

Functions $F_k(\xi_1,\xi_2,t)$ can be obtained from the equations of diffusion of the tangential components of the magnetic field into the body. To derive these equations we take the derivative of Eqs. (9a) and (9b) with respect to η and then couple it with Eqs. (10b) and (10a), respectively. As a result, the following equations are obtained

$$\frac{\partial H_{\xi_k}}{\partial t} = \frac{\partial^2 H_{\xi_k}}{\partial \eta^2} - a\frac{\partial^2 H_{\xi_k}}{\partial t^2} \qquad k=1,2 \tag{13}$$

Eqs. (13) are so-called "telegraph" equations that should be supplemented by the following conditions:

$$\eta=0: H_{\xi_k} = H^0_{\xi_k}(\xi_1,\xi_2,t); \qquad \eta\to\infty: H_{\xi_k}\to 0;$$
$$t=0:\ \vec{H}=0;\ \partial\vec{H}/\partial t = 0; \tag{14}$$

By using Laplace's transformation the following equations are obtained from (13)

$$(\overline{H}_{\xi_k})''_{\eta\eta} - (s+s^2a)\overline{H}_{\xi_k} = 0 \tag{15}$$

where

$$\overline{H}_{\xi_k}(\xi_1,\xi_2,\eta,s) = \int_0^\infty \exp(-st)H_{\xi_k}(\xi_1,\xi_2,\eta,t)dt$$

The solutions of (15) are written in the form

$$\overline{H}_{\xi_k} = \overline{H}^0_{\xi_k}\exp(-\eta\sqrt{s+s^2a}) \tag{16}$$

By integrating (16) with respect to η, one obtain

$$\overline{F}_k = \overline{H}^0_{\xi_k}\int_0^\infty \exp(-\eta\sqrt{s+s^2a})d\eta = \frac{\overline{H}^0_{\xi_k}}{\sqrt{s+s^2a}} \tag{17}$$

Inverse transformation of (17) yields:

$$F_k = a^{-1/2}\int_0^t H^0_{\xi_k}\exp\left(-\frac{t-t'}{2a}\right)I_0\left(\frac{t-t'}{2a}\right)dt' \tag{18}$$

Substitution of (18) in (11) and (12) gives the desired boundary conditions

$$\vec{n}\times\vec{E}^0 = p\frac{\partial}{\partial t}L[\vec{n}\times\vec{H}^0]\times\vec{n} \quad (19)$$

$$\vec{n}\cdot\vec{H}^0 = pL[Div\vec{H}^0] \quad (20)$$

where the time-operator L and surface divergence Div are defined as follows

$$L[f(\vec{r},t)] = a^{-1/2}\int_0^t f(\vec{r},t')\exp\left(-\frac{t-t'}{2a}\right)I_0\left(\frac{t-t'}{2a}\right)dt'$$

$$Div\vec{f} = div\left((\vec{n}\times\vec{f})\times\vec{n}\right)$$

The operator L describes the back impact of the field penetrating inside the body on the distribution of the incident field along the body's surface. This process has inductive nature since it caused by changes of the magnetic flux through the body due to the diffusion of the magnetic field into the body. The function $R(t)$

$$R(t) = a^{-1/2}\exp(-t/(2a))I_0(t/(2a))$$

gives the reaction of the field on the incident δ-pulse (Dirac function). Fig. 1 shows the distributions of function $L[f(t)]$ for several values of the parameter a and incident function $f(t)$.

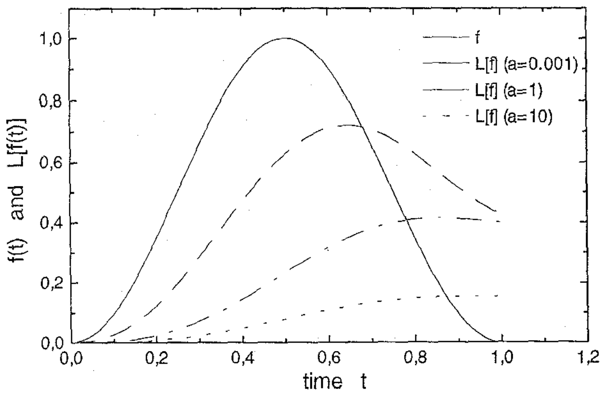

Fig. 1 The distributions of the impact function $L[f(t)]$, calculated for several values of the parameter a, and incident function $f(t)$

III. Surface Integral Equation Formulations

We start from the following standard space-time domain integral equations for the tangential components of the magnetic field (MFIE) and the electric field (EFIE) on the body's surface [7]

$$\vec{n}'\times\vec{H}^0(\vec{r},t) = 2\vec{n}'\times\vec{H}^{inc}(\vec{r},t) + \frac{\vec{n}'\times}{2\pi}\iint_s\left\{\frac{\varepsilon_0}{R}\frac{\partial}{\partial\tau}(\vec{n}'\times\vec{E}^0(\vec{r}',\tau)) + (\vec{n}'\times\vec{H}^0(\vec{r}',\tau))\times\frac{\vec{R}}{R^3} + \left(\vec{n}'\times\frac{\partial\vec{H}^0(\vec{r},\tau)}{\partial\tau}\right)\times\frac{\vec{R}}{cR^2} + (\vec{n}'\cdot\vec{H}^0(\vec{r}',\tau))\frac{\vec{R}}{R^3} + \left(\vec{n}'\cdot\frac{\partial\vec{H}^0(\vec{r}',\tau)}{\partial\tau}\right)\frac{\vec{R}}{cR^2}\right\}_{\tau=t-R/c} ds \quad (21)$$

$$\vec{n}'\times\vec{E}^0(\vec{r},t) = 2\vec{n}'\times\vec{E}^{inc}(\vec{r},t) - \frac{\vec{n}'\times}{2\pi}\iint_s\left\{\frac{\mu_0}{R}\frac{\partial}{\partial\tau}(\vec{n}'\times\vec{H}^0(\vec{r}',\tau)) - (\vec{n}'\times\vec{E}^0(\vec{r}',\tau))\times\frac{\vec{R}}{R^3} - \left(\vec{n}'\times\frac{\partial\vec{E}^0(\vec{r}',\tau)}{\partial\tau}\right)\times\frac{\vec{R}}{cR^2} - (\vec{n}'\cdot\vec{E}^0(\vec{r}',\tau))\frac{\vec{R}}{R^3} - \left(\vec{n}'\cdot\frac{\partial\vec{E}^0(\vec{r}',\tau)}{\partial\tau}\right)\frac{\vec{R}}{cR^2}\right\}_{\tau=t-R/c} ds \quad (22)$$

where $\vec{E}^{inc},\vec{H}^{inc}$ are the incident electric and magnetic fields, respectively; $\vec{R}=\vec{r}-\vec{r}'$. The vector $\vec{n}'=-\vec{n}$ is directed into the dielectric space normal to the conductor surface S. Re-writing (21)-(22) in the dimensionless variables in (6) and substituting the boundary conditions (19) and (20), the MFIE and EFIE become:

$$\vec{n}'\times\vec{H}^0 = 2\vec{n}'\times\vec{H}^{inc} + \frac{\vec{n}'\times}{2\pi}\iint_s\left\{-q^2pR^{-1}\frac{\partial^2L[\vec{H}^0]}{\partial\tau^2} + (\vec{n}'\times\vec{H}^0)\times\frac{\vec{R}}{R^3} + q\left(\vec{n}'\times\frac{\partial\vec{H}^0}{\partial\tau}\right)\times\frac{\vec{R}}{R^2} - \frac{p\vec{R}}{R^3}L[Div\vec{H}^0] - qp\frac{\vec{R}}{R^2}\frac{\partial}{\partial\tau}L[Div\vec{H}^0]\right\}_{\tau=t-qR} ds \quad (23)$$

$$-p\frac{\partial L[\vec{H}^0]}{\partial t} = 2\vec{n}'\times\vec{E}^{inc} - \frac{\vec{n}'\times}{2\pi}\iint_s\left\{R^{-1}\frac{\partial}{\partial\tau}(\vec{n}'\times\vec{H}^0) + p\frac{\partial L[\vec{H}^0]}{\partial\tau}\times\frac{\vec{R}}{R^3} + qp\frac{\partial^2L[\vec{H}^0]}{\partial\tau^2}\times\frac{\vec{R}}{R^2} - (\vec{n}'\cdot\vec{E}^0)\frac{\vec{R}}{R^3} - q\left(\vec{n}'\cdot\frac{\partial\vec{E}^0}{\partial\tau}\right)\frac{\vec{R}}{R^2}\right\}_{\tau=t-qR} ds \quad (24)$$

Note that (23) requires to use both conditions (19)-(20), whereas in the case of (24) only the condition (19) should be used. From Maxwell's equations for free space

$$\vec{n}'\cdot\vec{E}^0 = q^{-2}\int_0^t Div(\vec{n}'\times\vec{H}^0)dt \quad (25)$$

By substituting (25) into (24), introducing the equivalent surface electric current $\vec{J}^s=\vec{n}'\times\vec{H}^0$ and neglecting the terms of the order $O(p^2)$, Eqs.(23)-(24) can be written as follows

$$\vec{J}^s = 2\vec{n}'\times\vec{H}^{inc} + \frac{1}{2\pi}\vec{n}'\times\iint_S\left\{\vec{J}^s\times\frac{\vec{R}}{R^3} + q\frac{\partial\vec{J}^s}{\partial\tau}\times\frac{\vec{R}}{R^2}\right\}_{\tau=t-qR} ds + \frac{p}{2\pi}\vec{n}'\times\iint_S\frac{\vec{R}}{R^3}\left\{Div\Psi + qRDiv\frac{\partial\Psi}{\partial\tau}\right\}_{\tau=t-qR} ds -$$

$$-\frac{pq^2}{2\pi}\iint_S \left\{R^{-1}\frac{\partial^2}{\partial\tau^2}L[\vec{J}^s]\right\}_{\tau=t-qR} ds \qquad (26)$$

$$p\vec{n}'\times\frac{\partial}{\partial t}L[\vec{J}^s] = 2\vec{n}'\times\vec{E}^{inc} - \frac{1}{2\pi}\vec{n}'\times\iint_S \left\{R^{-1}\frac{\partial\vec{J}^s}{\partial\tau} - q^{-1}\frac{\vec{R}}{R^2}Div\vec{J}^s - q^{-2}\frac{\vec{R}}{R^3}\int_0^{\tau} Div\vec{J}^s d\tau'\right\}_{\tau=t-qR} ds + \frac{p}{2\pi}\iint_S \frac{\vec{n}'\cdot\vec{R}}{R^3}\frac{\partial}{\partial\tau}\left\{\Psi + qR\frac{\partial\Psi}{\partial\tau}\right\}_{\tau=t-qR} ds \qquad (27)$$

where $\Psi = L[\vec{n}'\times\vec{J}^s]$. If we omit terms containing p, then Eqs. (26) and (27) reduce to the usual equations for the scattering from the perfect conductor.

IV. Numerical Example

To illustrate the theory, we consider the problem of transient scattering from infinitely long cylinder of elliptical cross section. For the discussions of the numerical solution of the integral equations obtained above, we restrict ourselves by considering the TE case only (Fig. 2). Under these conditions, the vector $\vec{J}^s$ has only one component and can be treated as scalar. The following scale factors were used: the small semiaxis of ellips as D, the incident pulse duration as τ_p and $J^{inc}_{\max}D$ as I; here $J^{inc}_{\max}$ is the maximum of J^{inc}. The width of the incident pulse was taken to be D, i.e. q=1. The parameter a is equal to 1. The time t=0 corresponds to the time that the pulse reaches the body.

Fig. 2 shows the solutions of the problem using the perfect conductor limit (dotted line), the TIBC for good conductors used in [6], [8] (dashed line) and using the proposed TIBC for lossy dielectrics (solid line) for time t=1.5 and for p=0.2.

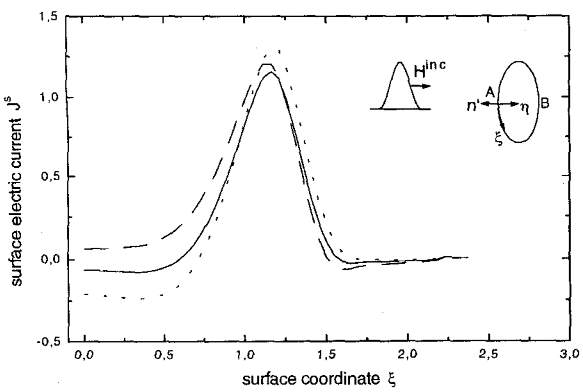

Fig. 2 The distributions of the surface electric current obtained using the perfect conductor limit (dotted line), the TIBC for good conductor (dashed line) and the TIBC for lossy dielectric (solid line) along a half of the cross section contour of long cylinder of elliptical cross section. The coordinate ξ has its origin at the point A and proceeds around the contour, ending at the point B.

As can been seen from the Fig.2, the correction allows the accuracy of the calculations to be of 10-15% higher as compared with the solution in the approximation for perfect or good conductors. Moreover, Fig. 2 shows the redistribution of the current along the body surface induced by the field diffusion into the body. This process can not be taken into account in the perfect conductor limit and its effect is to smooth out nonuniformities in the surface current distribution.

V. Conclusions

This paper presents new surface integral equation formulations to calculate transient scattering from a homogeneous body (conductor or lossy dielectric) when the electromagnetic field diffusion into the body should be taken into account. The problem is formulated in dimensionless variables related to the scatterer's surface in such a way, that the small parameter p, which is proportional to the ratio of the transient penetration depth and the characteristic size of the body, appears in the field equations. Using the perturbation techniques in the small parameter p, the impedance boundary condions for the normal component of the magnetic field and the tangential component of the electric field on the body surface are derived for transient incident fields. It was shown, that the boundary conditions are valid within the error $O(p^2)$. The use of the conditions together with space-time domain surface integral equations is proposed. As a result, the electric field and magnetic field integral equations over the scatterer's surface are decoupled and can be solved independently. A numerical example using the boundary conditions together with the surface integral equations is considered.

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