High Order Surface Impedance Boundary Conditions for the FDTD Method

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***Abstract*—Surface impedance boundary conditions (SIBC) of high order of approximation are implemented for 2-D FDTD formulation. The original FDTD equations for the boundary cell are represented in the series of equations similar in the left hand side. This approach takes into account the curvature of the body surface and tangential variation of the fields along the surface. It is shown that implementation of the high order SIBC does not cause new computational difficulties as compared to the well-known Leontovich's impedance boundary conditions.**

***Index Terms*—FDTD, numerical methods, perturbation techniques, scattering, surface impedance boundary conditions.**

I. INTRODUCTION

THERE have been a number of modifications to the traditional finite difference time domain (FDTD) especially in the last ten years, which avoids the limitations of the original paper [1]. One of them is implementation of the surface impedance boundary conditions (SIBCs) to remove the body of high conductivity from the computational space and replace it with a relationship between the tangential electric and magnetic fields on the surface of the body. Since the discretization size of the space in the FDTD lattice must be proportional to the wavelength propagating in the medium, removal of the medium, where the wavelength becomes very small, significantly simplifies the problem and consequently decreases the requirements of the computational resources.

This methodology has been successfully applied to FDTD for a variety of problems [2]–[4]. However, SIBCs presently used in FDTD codes are of the order of Leontovich's approximation in which the body's surface is considered as a plane and the field diffusion is taken into account only in the direction normal to the surface. As a result, the following important factors are neglected, namely: the curvature of the body's surface and the field diffusion in the tangential direction. Time domain SIBCs of high order of approximation allowing for these factors have been recently developed and implemented for the boundary element method [5], [6].

The purpose of this work is efficient implementation of high order SIBCs to FDTD method in order to improve an accuracy of the results without significant increase of the computational expenses as compared to low order SIBCs.

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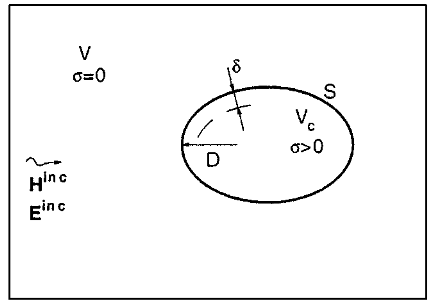

Fig. 1. Geometry of the problem.

II. SURFACE IMPEDANCE CONCEPT

Consider an electromagnetic pulse traveling in nonconducting space toward the homogeneous lossy dielectric body. Without any loss of generality we restrict ourselves by consideration of 2-D problem shown in Fig. 1. Then distribution of the field in both regions is described by Maxwell's equations:

Non-Conducting Region V:

$$\partial H_y/\partial t = -\mu_0^{-1}(\partial E_x/\partial z - \partial E_z/\partial x) \tag{1a}$$

$$\partial E_x/\partial t = -\varepsilon_0^{-1}\partial H_y/\partial z; \qquad \partial E_z/\partial t = \varepsilon_0^{-1}\partial H_y/\partial x. \tag{1b}$$

Conducting Region V_c:

$$\partial H_y/\partial t = -\mu^{-1}(\partial E_x/\partial z - \partial E_z/\partial x) \tag{2a}$$

$$E_x = -\sigma^{-1}\partial H_y/\partial z; \qquad E_z = \sigma^{-1}\partial H_y/\partial x. \tag{2b}$$

Let the incident pulse duration τ be such that electromagnetic penetration depth δ into the body remains small as compared with characteristic size D of the body:

$$\delta = (\tau/(\sigma\mu))^{1/2} \ll D. \tag{3}$$

Then (2a) and (2b) can be solved by analytical methods under the boundary layer approximation [5]. As a result, the following time domain high order surface impedance boundary conditions can be written and used instead of (2a) and (2b):

$$E_{\tan} = (E_{\tan})_1 + (E_{\tan})_2 + (E_{\tan})_3 = T_1 * H_y - \frac{1}{R} T_2 * H_y - T_3 * \left(\frac{H_y}{8R^2} + \frac{1}{2}\frac{\partial^2 H_y}{\partial s^2}\right). \tag{4}$$

Here "$*$" denotes the time-convolution product, R is the local radius of the curvature, s is the arc length, and time-dependant functions T_n, $n = 1, 2, 3$ are written in the following form:

$$T_1(t) = \left(\frac{\mu}{\varepsilon}\right)^{1/2}\left(\delta(t)+\frac{\sigma}{2\varepsilon}\left(I_1\left(\frac{\sigma t}{2\varepsilon}\right)-I_0\left(\frac{\sigma t}{2\varepsilon}\right)\right)\exp\left(-\frac{\sigma t}{2\varepsilon}\right)\right) \tag{5a}$$

$$T_2(t) = \varepsilon^{-1}\,\exp\left(-\frac{\sigma t}{\varepsilon}\right)\Bigg) \tag{5b}$$

$$T_3(t) = t(\varepsilon\sqrt{\varepsilon\mu})^{-1}\left(I_0\left(\frac{\sigma t}{2\varepsilon}\right)-I_1\left(\frac{\sigma t}{2\varepsilon}\right)\right)\exp\left(-\frac{\sigma t}{2\varepsilon}\right) \tag{5c}$$

where $I_n(x)$ is the modified Bessel function of order n and $\delta(t)$ is the Dirac function. The first term in (4) is the Leontovich SIBC and other terms give corrections of the order of the Mitzner and Rytov approximations, respectively.

Basic property of the SIBC in (4) is the following:

$$(E_{\tan})_2/(E_{\tan})_1 \ll O(p); \qquad (E_{\tan})_3/(E_{\tan})_1 \ll O(p^2)$$
$$p = \delta/D \ll 1 \tag{6}$$

where p is small parameter since it is equal to the ratio of the skin depth and characteristic size of the body's surface.

III. FDTD Formulation

Since the high order SIBC takes into account curvature of the body it is natural to use generalized FDTD for curved surfaces (contour path method [7]). In this method boundary cells adjacent to the surface of the scatterer are distorted in order to conform to the arbitrary curved surface of the scatterer. Fig. 1 shows a nonrectangular boundary cell for the generalized FDTD formulation. The magnetic field updating equation can be written in the following form:

$$\begin{aligned}&A.H_y^{n+1/2}(i-1/2,\, k-1/2)\\ &\quad= A.H_y^{n-1/2}(i-1/2,\, k-1/2)\\ &\qquad+\frac{\Delta t}{\mu_1}\begin{pmatrix} gE_z^n(i,\, k-1/2)-fE_z^n(i-1,\, k-1/2)\\ -sE_{\tan}^n(i-1/2,\, k)+hE_x^n(i-1/2,\, k-1)\end{pmatrix}\end{aligned} \tag{7}$$

where g, f, s, h are the side lengths of the distorted cell and A is the area of the patch.

Substituting SIBC (4) into (7) one obtains

$$\begin{aligned}&AH_y^{n+1/2}(i-1/2,\, k-1/2)\\ &\quad= AH_y^{n-1/2}(i-1/2,\, k-1/2)\\ &\qquad+\frac{\Delta t}{\mu}\Bigg\{gE_z^n(i,\, k-1/2)-fE_z^n(i-1,\, k-1/2)\\ &\qquad\quad -hE_x^n(i-1/2,\, k-1)-sH_y^n(i-1/2,\, k-1/2)\\ &\qquad\quad *\left[T_1-\frac{1}{R}T_2-\frac{1}{8R^3}T_3\right]\\ &\qquad\quad +\frac{1}{2}\,\frac{\partial^2 H_y^n(i-1/2,\, k-1/2)}{\partial s^2}*T_3\Bigg\}.\end{aligned} \tag{8}$$

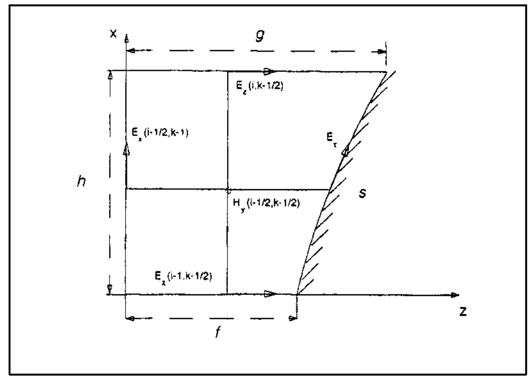

Fig. 2. FDTD boundary cell.

To approximate the term $\partial^2 H_y/\partial s^2$ in discrete space, the following representation is used:

$$\begin{aligned}\frac{\partial^2 H_y^n}{\partial s^2} &= \frac{\partial^2 H_y^n}{\partial x^2}\left(\frac{\partial s}{\partial x}\right)^{-1}+\frac{\partial^2 H_y^n}{\partial y^2}\left(\frac{\partial s}{\partial y}\right)^{-1}\\ &\quad+\frac{\partial^2 H_y^n}{\partial x\partial y}\left[\left(\frac{\partial s}{\partial x}\right)^{-1}+\left(\frac{\partial s}{\partial y}\right)^{-1}\right]\\ &\quad-\Bigg\{\frac{\partial H_y^n}{\partial x}\,\frac{\partial^2 s}{\partial x^2}\left(\frac{\partial s}{\partial x}\right)^{-2}+\frac{\partial H_y^n}{\partial y}\,\frac{\partial^2 s}{\partial x\partial y}\left(\frac{\partial s}{\partial y}\right)^{-2}\\ &\qquad+\frac{\partial H_y^n}{\partial x}\,\frac{\partial^2 s}{\partial x\partial y}\left(\frac{\partial s}{\partial x}\right)^{-2}\\ &\qquad+\frac{\partial H_y^n}{\partial y}\,\frac{\partial^2 s}{\partial y^2}\left(\frac{\partial s}{\partial y}\right)^{-2}\Bigg\}.\end{aligned}$$

Notice that quantities H_y^n in (8) is time indexed at time step n so that the following approximation has to be done:

$$H_y^n \approx \tfrac{1}{2}(H_y^{n+1/2}+H_y^{n-1/2}). \tag{9}$$

As can be seen from (8) and (9) the FDTD update equations for boundary cells are in implicit form and not suitable for computational purposes. However, property (6) makes it possible to reduce (8) using the perturbation technique to a series of equations that allow proceeding the numerical computation on the boundary in step by step manner.

IV. Perturbation Technique

Following theory of the perturbation methods, we introduce the basic scale factors I, D and τ for the current, Cartesian coordinates and time, respectively. Then we transfer to the following nondimensional variables [5], [8]:

$$\tilde{H} = HD/I; \qquad \tilde{E} = E\tau/(\mu I). \tag{10}$$

Here and below sign "$\sim$" denotes nondimensional values.

With variables in (10), (7) takes the form:

$$\begin{aligned}&\tilde{H}_y^{n+1/2}(i-1/2,\, k-1/2)\\ &\quad= \tilde{H}_y^{n-1/2}(i-1/2,\, k-1/2)\\ &\qquad+\frac{q}{Q}\begin{pmatrix}\tilde{g}\tilde{E}_z^n(i,\, k-1/2)-\tilde{f}\tilde{E}_z^n(i-1,\, k-1/2)\\ -\tilde{s}\tilde{E}_{\tan}^n(i-1/2,\, k)+\tilde{h}\tilde{E}_x^n(i-1/2,\, k-1)\end{pmatrix}.\end{aligned} \tag{11}$$

Here Q is the Courant number and q is the electric dimension of the body defined as

$$q = D/(c\tau)$$

where c is the light velocity. Values $g,\ f,\ s,$ and h in (11) have been normalized by scale factor

$$A/\Delta; \qquad \Delta = \Delta x \Delta z \Big/ \sqrt{\Delta x^2 + \Delta y^2}$$

where Δx and Δz are rectangular cell sizes.

Taking into account (10), boundary condition (4) can be represented in the following nondimensional form:

$$\begin{aligned}\tilde{E}_{\tan} &= (\tilde{E}_{\tan})_1 + p(\tilde{E}_{\tan})_2 + p^2(\tilde{E}_{\tan})_3 \\ &= \tilde{T}_1 * \tilde{H}_y^n - p\,\frac{1}{\tilde{R}}\,\tilde{T}_2 * \tilde{H}_y^n - p^2 \tilde{T}_3 * \left(\frac{\tilde{H}_y^n}{8\tilde{R}^2} + \frac{1}{2}\,\frac{\partial^2 \tilde{H}_y^n}{\partial \tilde{s}^2}\right) \\ &= \tilde{T}_1 * \frac{\tilde{H}_y^{n+1/2} + \tilde{H}_y^{n-1/2}}{2} - p\,\frac{1}{\tilde{R}}\,\tilde{T}_2 * \frac{\tilde{H}_y^{n+1/2} + \tilde{H}_y^{n-1/2}}{2} \\ &\quad - p^2 \tilde{T}_3 * \left(\frac{\tilde{H}_y^{n+1/2}}{16\tilde{R}^2} + \frac{\tilde{H}_y^{n-1/2}}{16\tilde{R}^2} + \frac{1}{4}\,\frac{\partial^2 \tilde{H}_y^{n+1/2}}{\partial \tilde{s}^2} \right. \\ &\quad \left. + \frac{1}{4}\,\frac{\partial^2 \tilde{H}_y^{n-1/2}}{\partial \tilde{s}^2}\right). \end{aligned} \tag{12}$$

Represent the magnetic field as power series in the small parameter p:

$$\begin{aligned}\tilde{H}_y^{n+1/2}&(i-1/2,\, k-1/2) \\ &= \tilde{H}_{y1}^{n+1/2} + p\tilde{H}_{y2}^{n+1/2} + p^2 \tilde{H}_{y3}^{n+1/2}. \end{aligned}\tag{13}$$

Substituting representations (12) and (13) into (11) and equating the coefficients of equal powers of p, the following equations for terms of expansions (13) are obtained:

p^0:

$$\begin{aligned}\tilde{H}_{y1}^{n+1/2} &+ \frac{q}{2Q}\, s\tilde{T}_1 * \tilde{H}_{y1}^{n+1/2} \\ &= \tilde{H}_y^{n-1/2} + \frac{q}{Q}\left[\tilde{g}\tilde{E}_z^n(i,\, k-1/2) - \tilde{f}\tilde{E}_z^n(i-1,\, k-1/2)\right. \\ &\quad \left. - \frac{\tilde{s}}{2}\,\tilde{T}_1 * \tilde{H}_y^{n-1/2} + \tilde{h}\tilde{E}_x^n(i-1/2,\, k-1)\right]\end{aligned} \tag{14}$$

p^1:

$$\tilde{H}_{y2}^{n+1/2} + \frac{q}{2Q}\, s\tilde{T}_1 * \tilde{H}_2^{n+1/2} = \frac{q}{2Q}\,\frac{\tilde{s}}{\tilde{R}}\,\tilde{T}_2 * (\tilde{H}_{y1}^{n+1/2} + \tilde{H}_y^{n-1/2}) \tag{15}$$

p^2:

$$\begin{aligned}\tilde{H}_{y3}^{n+1/2} &+ \frac{q}{2Q}\, s\tilde{T}_1 * \tilde{H}_3^{n+1/2} \\ &= \frac{q}{2Q}\,\frac{\tilde{s}}{\tilde{R}}\,\tilde{T}_2 * \tilde{H}_{y2}^{n+1/2} - \frac{q}{4Q}\,\tilde{T}_3 \\ &\quad * \left(\frac{\tilde{H}_{y1}^{n+1/2}}{4\tilde{R}^2} + \frac{\tilde{H}_y^{n-1/2}}{4\tilde{R}^2} + \frac{\partial^2 \tilde{H}_{y1}^{n+1/2}}{\partial \tilde{s}^2} + \frac{\partial^2 \tilde{H}_y^{n-1/2}}{\partial \tilde{s}^2}\right)\end{aligned} \tag{16}$$

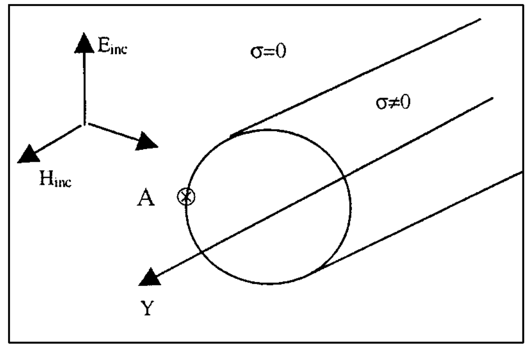

Fig. 3. Plane wave on a conducting cylinder.

where the following notation has been used:

$$\tilde{H}_y^{n-1/2} = \tilde{H}_y^{n-1/2}(i-1/2,\, k-1/2).$$

Finally, returning to dimensional variables, the proposed formulation is written in the form:

$$H_y^{n+1/2}(i-1/2,\, k-1/2) = H_{y1}^{n+1/2} + H_{y2}^{n+1/2} + H_{y3}^{n+1/2} \tag{17}$$

$$\begin{aligned}H_{y1}^{n+1/2} &+ \frac{\Delta t}{\mu A} sT_1 * H_{y1}^{n+1/2} \\ &= H_y^{n-1/2} + \frac{\Delta t}{\mu A}\left[gE_z^n(i,\, k-1/2) - fE_z^n(i-1,\, k-1/2)\right. \\ &\quad \left. - \frac{s}{2}\, T_1 * H_y^{n-1/2} + hE_x^n(i-1/2,\, k-1)\right]\end{aligned} \tag{18}$$

$$\begin{aligned}H_{y2}^{n+1/2} &+ \frac{\Delta t}{2\mu A}\, sT_1 * H_2^{n+1/2} \\ &= \frac{\Delta t}{2\mu A}\,\frac{s}{R} T_2 * (H_{y1}^{n+1/2} + H_y^{n-1/2})\end{aligned} \tag{19}$$

$$\begin{aligned}H_{y3}^{n+1/2} &+ \frac{\Delta t}{2\mu A}\, sT_1 * H_3^{n+1/2} \\ &= \frac{\Delta t}{2\mu A}\,\frac{s}{R}\, T_2 * H_{y2}^{n+1/2} - \frac{\Delta t}{4\mu A}\, T_3 \\ &\quad * \left(\frac{H_{y1}^{n+1/2}}{4R^2} + \frac{H_y^{n-1/2}}{4R^2} + \frac{\partial^2 H_{y1}^{n+1/2}}{\partial s^2} + \frac{\partial^2 H_y^{n-1/2}}{\partial s^2}\right).\end{aligned} \tag{20}$$

The first equation gives implementation of the Leontovich SIBC. The second and third equations give the correction by taking into account curvature of the body's surface and the field variation in the tangential direction. Emphasize that (18)–(20) are different only in the right hand sides.

V. Numerical Example

To validate the formulation of the high order SIBC the following numerical example is conducted using an 2-D FDTD code. An incident TE_y plane wave with the following material

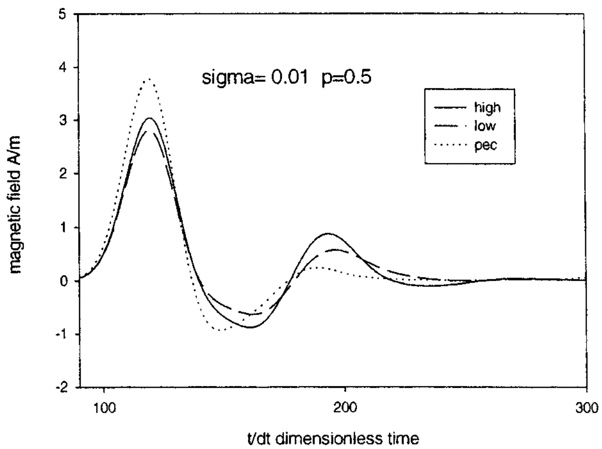

Fig. 4. Total magnetic field at point A.

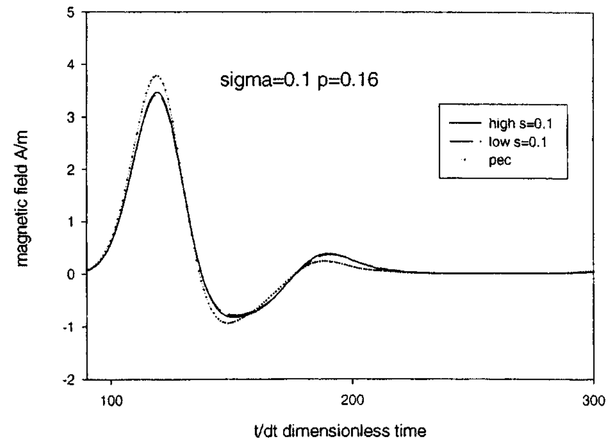

Fig. 5. Total magnetic field at point A.

parameters of an infinitely long conducting cylinder along the y axis is considered in the scattered field formulation (Fig. 3):

$$\varepsilon_1 = \varepsilon_2 = \varepsilon_0; \quad \mu_1 = \mu_2 = \mu_0; \quad \sigma_1 = 0; \quad \sigma_2 \neq 0;$$

$$\text{Excitation} = 1000 \exp\left(-16\times\left(\frac{t}{7.071\times 10^{-8}} - 1\right)^2\right).$$

The following mesh parameters were used:

$$\delta z = 50 \text{ cm}; \qquad \delta t = 1.178 \text{ nsec}$$

$$N_{z-space\,steps} = 100; \qquad N_{time\,steps} = 300.$$

The diameter of the cylinder is 12 cells which is almost 1/3 of the average wave length of the Gaussian pulse ($\lambda = 3.10^8\tau/2$). τ is the duration of the Gaussian pulse. The distorted cells are used adjacent to the cylinder surface to conform FDTD cells to the curved surface. The total magnetic field at point A (Fig. 3) is observed and shown for low order and high order approximation and compared with each other for different values of small parameter.

According to [9] the selection of approximation order should be based on the small parameter value in the following ranges:

PEC conditions	$p < 0.06$;
Low order conditions	$0.06 < p < 0.25$;
High order conditions	$0.25 < p$.

The agreement with the above range can be seen in the results. In Fig. 3 the difference between low and high order is significant since the small parameter is in the range of high order but when it is in the range of low order in Fig. 4 the difference is negligible. The next step in this work is to perform detailed analysis of limitations of the proposed method by comparison with the results obtained using the FDTD without an approximation.

VI. Conclusion

The combination of FDTD with high order SIBC is used to solve transient scattering from body of finite conductivity. This technique, which had been successfully used in low order SIBC for FDTD and high order SIBC for boundary element method, is proved to be a good tool to remove the conducting body from the computation domain. This permits to use of smaller cell size, which leads to low computational cost. In addition to standard implementation of high order SIBC for FDTD, the perturbation technique is employed to transform implicit FDTD update equations on the boundary into series of equations similar to the low order SIBC for FDTD therefore the same approach can be used to efficiently solve the time convolution term.

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