Application of Biorthogonal Wavelets on the Interval [0,1] to 2D EM Scattering

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Abstract-**A biorthogonal wavelet-based MoM is presented. Specifically, the MoM based on compactly supported biorthogonal spline wavelets on the interval [0,1] are implemented and applied to solving 2-Dimensional electromagnetic scattering problems. The numerical results show the effectiveness and usefulness of the proposed approach.**

Index terms- **Biorthogonal wavelets, moment method, electromagnetic scattering, sparse matrix.**

I. INTRODUCTION

Wavelets have found considerable use in the representation of signals and data compression. This is because wavelets have some important and useful features. One is their time (space) and frequency localization. Another property is called cancellation property, or zero-moments. From the numerical analysis point of view, wavelets preserve the compact support property of traditional basis functions. When dealing with differential operators compactly supported bases give rise to sparse matrices, while when dealing with integral equations, the cancellation property of wavelets, together with their compact support, results in nearly sparse matrices. This is one of the advantages of using wavelets as basis functions over the traditional bases.

Wavelets have been used to solve partial differential and integral equations in electromagnetics [2-8], and "sparse" matrices were obtained in all cases. Whole-line or periodized orthogonal wavelet bases were used to solve integral equations with bounded domains in EM scattering and guided waves [2-6]. There were difficulties in treating domain boundaries when whole-line wavelet bases were used. Orthogonal wavelets on the interval [0,1] for analysis of thin wire antennas and scatterers [8], and semiorthogonal wavelets on [0,1] for solving integral equations of the first kind [7], which overcame the difficulties mentioned above, have been presented. The focus in this paper is on the use of compactly supported biorthogonal wavelets on the interval [0,1] in the method of moments for the solution of the integral equations in EM scattering and guided wave problems. Biorthogonal wavelets have all the major features of orthogonal and semiorthogonal wavelets, but also offer additional properties. In the biorthogonal case, there are two related wavelet bases, namely primal wavelet bases and dual wavelet bases. The dual wavelet bases can have different smoothness than primal wavelet bases. Since the test bases and trial bases actually play different roles in the method of moments, biorthogonal wavelet bases provide flexibility for choosing proper trial bases and test bases. The result is improvement of solution efficiency and accuracy. Interestingly, in some cases one can construct a family of biorthogonal wavelet bases from a single scaling function. This wavelet family has a number of different orders of zero-moments for primal wavelets and different smoothness for the dual wavelets. The rest of this paper is organized as follows: In the next section we outline the major features of compactly supported biorthogonal spline wavelets on the interval [0,1] used in this work. In section III we briefly describe how the method of moments based on the biorthogonal wavelets is implemented for treatment of integral equations with curved and bounded integration paths. Numerical examples are given in section IV. 2D electromagnetic scattering is analyzed by solving the corresponding EFIE. A brief conclusion is given in section V.

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II. COMPACTLY SUPPORTED BIORTHOGONAL SPLINE WAVELET BASES ON THE INTERVAL [0,1]

Biorthogonal spline wavelets on [0,1] at a single resolution level are composed of a finite number of so-called boundary wavelets and interior wavelets [9,10]. A lowest resolution level needs to be prescribed for a specific biorthogonal wavelet pair for analysis to be realistic. The primal wavelet and dual wavelet bases are expressed as [10]

$$\Psi = \bigcup_{j=J_0-1}^{\infty} \Psi_j, \quad \tilde{\Psi} = \bigcup_{j=J_0-1}^{\infty} \tilde{\Psi}_j, \tag{1}$$

where Ψ_j and $\tilde{\Psi}_j$, are the sub-bases of the primal and dual wavelets at resolution level j, respectively, and are given by

$$\Psi_j = \{\Psi_j^L, \Psi_j^I, \Psi_j^R\} \quad \tilde{\Psi}_j = \{\tilde{\Psi}_j^L, \tilde{\Psi}_j^I, \tilde{\Psi}_j^R\} \tag{2}$$

with

$$\Psi_j^L = \{\psi_{j,k}^{left}, k = l_w - m_w,\ldots,l_w - 1\},$$

$$\Psi_j^R = \{\psi_{j,k}^{right}, k = 2^j + l_w - 2m_w,\ldots,2_j + l_w - m_w - 1\},$$

and $$\Psi_j^I = \{\psi_{j,k}^{interior}, k = l_w,\ldots,2^j + l_w - 2m_w - 1\},$$

the left boundary, right boundary, and interior wavelet sets on the primal side, and

$$\tilde{\Psi}_j^L = \{\tilde{\psi}_{j,k}^{left}, k = \tilde{l}_w - \tilde{m}_w,\ldots,\tilde{l}_w - 1\},$$

$$\tilde{\Psi}_j^R = \{\tilde{\psi}_{j,k}^{rightt}, k = 2^j + \tilde{l}_w - 2\tilde{m}_w,\ldots,2^j + \tilde{l}_w - \tilde{m}_w - 1\}$$

and $$\tilde{\Psi}_j^I = \{\tilde{\psi}_{j,k}^{interior}, k = \tilde{l}_w,\ldots 2^j + \tilde{l}_w - 2\tilde{m}_w - 1\},$$

the left boundary, right boundary, and interior wavelet sets on the dual side. m_w and $\tilde{m}_w$ are the number of the boundary wavelets on the primal and dual side, respectively, l_w and $\tilde{l}_w$ denote the first shift indices of the interior primal and dual wavelets (from left to right). Note that

$$\Psi_j = \Phi_{J_0} = \{\Phi_{J_0}^L, \Phi_{J_0}^I, \Phi_{J_0}^R\}, \quad j = J_0 - 1 \tag{3}$$

$$\tilde{\Psi}_j = \tilde{\Phi}_{J_0} = \{\tilde{\Phi}_{J_0}^L, \tilde{\Phi}_{J_0}^I, \tilde{\Phi}_{J_0}^R\}, \quad j = J_0 - 1 \tag{4}$$

where Φ_j and $\tilde{\Phi}_j$ account for the sub-bases of the primal and dual scaling functions on [0,1] at the lowest resolution level J_0. From a single cardinal B-spline function of order d, a family of biorthogonal wavelet bases with exactness order of $\tilde{d}$ for dual scaling functions, satisfying $\tilde{d} \geq d$ and $\tilde{d}+d = even$, can be constructed [9,10]. There are exactly d left and right boundary scaling functions on the primal side and $\tilde{d}$ boundary scaling functions at left and right boundaries on the dual side for a fixed pair of $(d,\tilde{d})$. Moreover, there is a fixed and same number of the total scaling functions on the primal and dual sides within the interval [0,1]. The numbers of the boundary wavelets, i.e., m_w and $\tilde{m}_w$, can be chosen to be fixed for a given pair of $(d,\tilde{d})$, while the number of the total wavelet functions over [0,1] at a given resolution level j is fixed, i.e., 2^j. The major features of the biorthogonal spline wavelet bases described above can be outlined as follows:

(i). The sub-basis Φ_j in (3) is exact of order d on [0,1], i.e., they generate all polynomials up to order d on [0,1]; Similarly the sub-basis $\tilde{\Phi}_j$ in (4) is exact of order $\tilde{d}$ on [0,1].

(ii). The two-scale equations are given in matrix form as

$$\Phi_j = M_{j,0}\Phi_{j+1},\quad \tilde{\Phi}_j = \tilde{M}_{j,0}\tilde{\Phi}_{j+1},\quad j \geq J_0 \tag{5}$$

$$\Psi_j = M_{j,1}\Phi_{j+1},\quad \tilde{\Psi}_j = \tilde{M}_{j,1}\tilde{\Phi}_{j+1},\quad j \geq J_0 \tag{6}$$

where $M_{j,0}, \tilde{M}_{j,0}, M_{j,1}$ and $\tilde{M}_{j,1}$ are the matrix form of dilation coefficients.

(iii). Biorthogonality relations:

$$\left(\Psi_j,\tilde{\Psi}_{j'}\right)_{[0,1]} = \delta_{j,j'} I^{(2^j)},\quad j,j' \geq J_0 - 1, \tag{7}$$

where $\left(\Psi_j,\tilde{\Psi}_{j'}\right)_{[0,1]}$ denotes a matrix-form inner product over [0,1], $I^{(2^j)}$ the unit matrix of size $2^j \times 2^j$, $\delta_{j,j'}$ the delta function.

(iv). Zero-moments:

$$\int_{[0,1]} x^\alpha \psi_{j,k} dx = 0,\quad \int_{[0,1]} x^\beta \tilde{\psi}_{j,k} dx = 0,\quad \alpha < \tilde{d}, \beta < d. \tag{8}$$

(v). Biorthogonal wavelet expansions:

Any function $f \in L_2(R([0,1]))$ has a unique expansion

$$f = \sum_{j=J_0-1} \sum_{k\in\nabla_j} \left\langle f, \tilde{\psi}_{j,k} \right\rangle_{[0,1]} \psi_{j,k} \tag{9}$$

or

$$f = \sum_{j=J_0-1} \sum_{k\in\nabla_j} \left\langle f, \psi_{j,k} \right\rangle_{[0,1]} \tilde{\psi}_{j,k} \tag{10}$$

where $\langle \cdot,\cdot \rangle_{[0,1]}$ is the inner product defined over [0,1], ∇_j denotes the shift index sets of the wavelets on [0,1]. The biorthogonal spline wavelet bases described above were implemented by object-oriented programming. Fig. 1 shows the graphs of the biorthogonal spline wavelets with d = 2, $\tilde{d}$ = 6 and J_0 = 5. Note that only half the dual boundary scaling functions is displayed for brevity. It should be pointed out that wavelets on the dual side have more smoothness, and smoother sets, equivalent to larger number of zero-moments of primal wavelets, can be obtained by choosing a larger value for $\tilde{d}$.

III. BIORTHOGONAL WAVELET-BASED MoM FOR THE SOLUTION OF 2D EFIE

The electric field integral equation (EFIE) for 2D EM scattering (TM case) is expressed as

$$E_z^{inc}(\rho) = \frac{k\eta}{4}\int_c J_z(\rho')H_0^{(2)}(k|\rho-\rho'|)dl' \tag{11}$$

where c denotes the contour representing the surface of the scatterer, ρ' and ρ account for the position vectors corresponding to source and field points on c, respectively, k and η are the free space wave number and wave impedance, $J_z(\cdot)$, the unknown current density on the object surface, $E_z^{inc}(\cdot)$ denotes the incident electric field component in z-direction, $H_0^{(2)}(\cdot)$, the Hankel function of the second kind and order zero. MoM approach to the solution of integral equations requires: (i) Determination of a proper trial basis for the unknown function expansion and a proper test basis for the realization of the weighting scheme; (ii) Efficient calculation of the matrix coefficients for the corresponding discretized systems of equations, which usually requires calculation of integrals involving singularity over the whole solution domain; (iii) An efficient solver for the matrix equations. The treatment related to these issues is discussed next.

A. Domain Transformations: For the biorthogonal wavelets on the interval [0,1] to be used as bases in MoM approach for the solution of integral equations like (11), the first requirement is to perform domain transformations so that the wavelet bases on [0,1] can be properly used to expand the unknown function, $J_z(\cdot)$ which are defined on a general

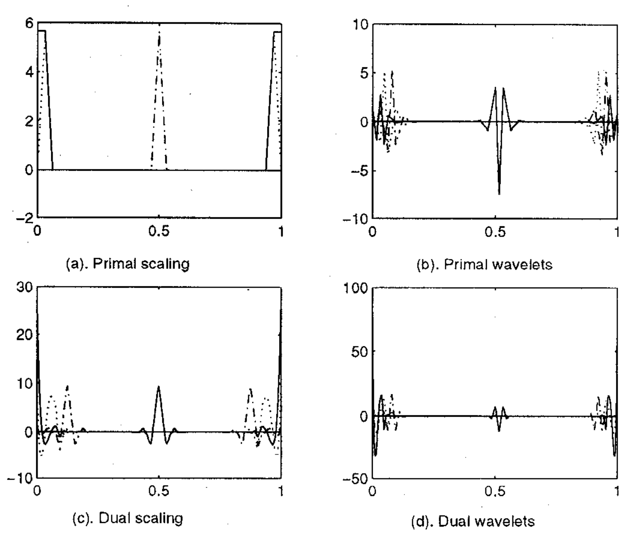

Fig. 1. Biorthogonal spline wavelet bases on the interval [0,1]

curve. Specifically we need to transform the domain consisting of a general curve c to [0,1], or, in other words, to transform bases on [0,1] to bases on the curve. This can be done by using a parametrization of the underlying curve and a two-step mapping. Denoting by ξ and ξ' the points in [0,1] corresponding to the field and source points on the curve c, after accomplishing the domain transformations, (11) can be rewritten as

$$E_z^{inc}(\rho(\xi)) = \frac{k\eta}{4}\int_0^1 J_z(\rho'(\xi'))H_0^{(2)}(k|\rho(\xi)-\rho'(\xi')|) \, |\omega(\xi')|d\xi', \quad (12)$$

where $|\omega(\xi')|$ is the integral scale determined by the specific two-step mapping involved.

B. Using Smoother Wavelets as Trial Basis and Coarser Wavelets as Test Basis: As mentioned in section I, dual wavelets have different smoothness than their counterpart on the primal side in a general biorthogonal wavelet framework. In MoM, the trial basis is used for the representation of unknown functions. Thus smoothness of the basis functions is generally required. The test basis is used for the realization of the weighting scheme and smoothness of the basis functions is usually not critical. Thus it is natural to choose the smoother wavelet basis as the expansion basis, the other as the testing basis. In this work the dual wavelets are chosen as expansion functions, the primal wavelets are used as test bases since the dual wavelets are smoother. The biorthogonal wavelet expansions of the surface current in (11) and (12) are written as

$$J_z(\rho'(\xi')) = \sum_{j=J_0-1}^{J_h}\sum_{k\in\nabla_j} J_{j,k}\tilde{\psi}_{j,k}(\xi') \quad \xi'\in[0,1], \quad (13)$$

where J_h is the pre-specified highest resolution level in the analysis. The MoM gives the following matrix equation

$$[A][I] = [G] \quad (14)$$

where the elements of the coefficient matrix A and the right hand-side vector G are given by

$$(A)_{i,l,j,k} = \frac{k\eta}{4}\int_0^1 \psi_{i,l}(\xi)|\omega(\xi)|d\xi \cdot \left[\int_0^1 \tilde{\psi}_{j,k}(\xi')H_0^{(2)}(k|\rho(\xi)-\rho'(\xi')|)|\omega(\xi')|d\xi'\right], \quad (15)$$

$$(G)_{i,l} = \int_0^1 \psi_{i,l}(\xi)E_z^{inc}(\rho(\xi))|\omega(\xi)|d\xi \cdot \quad (16)$$

with $i,j = J_0-1,\dots,J_h,\ l\in\nabla_i, k\in\nabla_j$.

C. Numerical Considerations in Computation of Coefficients: There have been two approaches to calculate the coefficients in (15) and (16). One relies on direct use of fast wavelet transform (FWT), which requires pre-expansion of the integral kernel containing singularity by using 2D wavelets. The other is the use of traditional numerical integration. In this work we chose the second approach. Although wavelet frameworks are very suited to dealing with singularities, for the current case of separable wavelet bases, this advantage does not seem so apparent and straightforward because the singularity in this kernel is closely related to cooperation of the two arguments, ξ and ξ', as can be seen from (15). On the other hand, using numerical integration (like Gaussian integration) can still take full advantage of important features of the biorthogonal spline wavelet bases because: first, all the wavelets have compact support on [0,1], so the integration in (15) and (16) does not need to be spread to the whole domain [0,1]; second, unlike traditional element-based basis functions, the wavelets are element-independent. This makes adaptive calculation of the integration near any singular point (quadrature points) possible. Efforts have been made toward this goal in this work. The numerical integration of (15) and (16) involve the evaluation of the wavelets at the quadrature points. In this work the wavelets on the primal side have explicit expressions, thus the evaluation of the wavelets poses no extra problems. There are no explicit expressions for the dual side wavelets, but their values at the dyadic points can be calculated exactly by means of a recursive procedure. The curves of the wavelets are pre-calculated at a sufficient number of dyadic points and stored. The values of the wavelets at any point can be obtained by interpolation. Moreover the values of the wavelets at the quadrature points can be precalculated and stored for subsequent use. Our tests showed that the time needed to calculate each element in (15) is about the same as that spent on computing each entry for the traditional bases if the traditional bases are polynomials of the same order as the wavelet bases used.

It is observed that the feature of cancellation and compact support of the wavelets makes the resulting matrix extremely "sparse". By this we mean that a large number of the coefficients are very small compared with the other coefficients. These very small coefficients correspond to wavelet elements whose supports do not overlap and are far away from each other. These coefficients have little influence on the final solutions of the discretized matrix equations, thus we can simply set them to zero, and a sparse matrix is obtained. A threshold parameter δ with $(0<\delta<1)$ is introduced and all elements smaller than the product of the largest element of the matrix with the specified threshold are set to zero. The sparse matrix can be efficiently solved by using any iterative method such as the conjugate gradient solver employed in this work.

IV. NUMERICAL EXAMPLES

The method of moments based on the biorthogonal spline wavelets on the interval [0,1] has been implemented by object-oriented programming and was used to analyze 2D electromagnetic scattering problems. Two examples are given here. The first is EM scattering from a circular conducting cylinder of radius $a = 0.1\lambda$ and the second is an open structure with $L = 0.1\lambda$ (Fig. 2). Both are illuminated by plane waves. In the analysis, the lowest resolution level was chosen to be $J_0 = 5$, while $J_h = 5,6$ were used as the highest resolution levels. There were 129 basis functions on both the primal and dual sides, with 33 scaling functions and 96 wavelets for the case $J_h = 6$. Fig. 3 shows the magnitudes and phases of the surface current on the cylinder from the direct solution of the original matrix and the solutions of the matrices by applying different thresholds. The results from the conventional MoM are also displayed. It can be seen that good agreement is achieved between these two methods. Similar results for the scatterer of the open structure are illustrated in Fig. 4. Fig. 5 shows the sparsity patterns of the coefficient matrices for the two examples with a threshold $\delta = 0.0001$. The variable nz in Fig. 5 is the number of the non-zero elements in the 129x129 matrices. Thus the percentages of the non-zero elements for the two examples are 34.6% and 30.7%, respectively. The time saving from solving the sparse matrices can be estimated by evaluating the operations performed in applying the conjugate gradient method. The sparseness of the resulting matrices will increase as the sizes of problems become larger and more levels of wavelets are used in the analysis.

V. CONCLUSIONS

An efficient method of moments has been implemented based on compactly supported biorthogonal spline wavelets on the interval [0,1]. This technique gives rise to a "sparse" matrix when dealing with EFIE from 2D EM scattering. This phenomenon will also exist in 3D. Its full advantages are especially apparent when dealing with large problems. The advantages of biorthogonal wavelets over other wavelet bases rely on the fact that they are more flexible in choosing proper trial and test bases thus the efficiency of solution improves.

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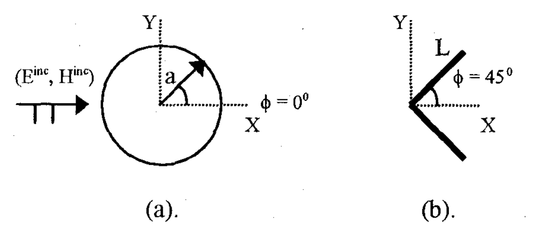

Fig. 2. Conducting cylinder (a) and an open structure (b) illuminated by a plane wave

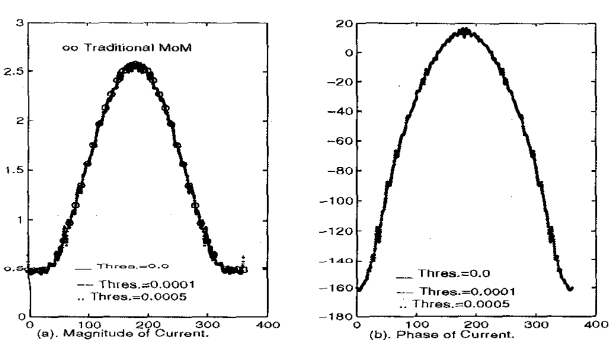

Fig. 3 Current and phase distributions along the Circular cylinder surface

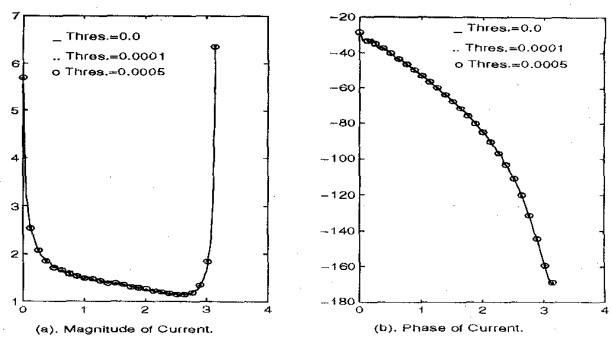

Fig. 4. Current and phase distributions along the open structure surface

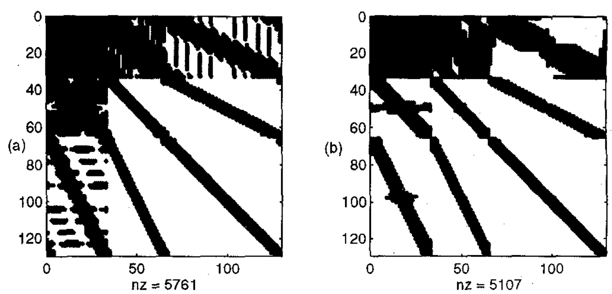

Fig. 5. Sparsity patterns of the 129x129 matrices for threshold = 0.0001.

REFERENCES

[1] A. G. Beylkin, R.R. Coifman and V. Rolhlin, "Fast wavelet transform and numerical algorithms I", Comm. Pure Appl. Math., 44, 1991, pp 141-183.

[2] B. Z. Steinberg and Y. Leviatan, "On the use of wavelet expansions in the method of moments," *IEEE Trans. Antennas Propagat.*, Vol. 41, pp. 610-619, May 1993.

[3] K. Sabetfakhri and L. P. B. Katehi, "Analysis of integrated millimeter-wave and submillimeter-wave waveguides using orthonormal wavelet expansions", *IEEE Trans. Microwave Theory and Tech.*, Vol. 42, Dec. 1994, pp2412-2422.

[4] G. Wang and G. Pan, "Full wave analysis of microstrip floating line structures by wavelet expansion method," IEEE Trans. Microwave Theory and Tech., Vol. 43, Jan. 1995, pp. 131-142.

[5] G. Wang, "A hybrid wavelet expansion and boundary element analysis of electromagnetic scattering from conducting objects," *IEEE Trans. Antennas Propagat.*, Vol. 43, Feb. 1995, pp. 170-178.

[6] G. Wang and G. Pan, "A hybrid wavelet expansion and boundary element analysis for multiconductor transmission lines in multilayered dielectric media", IEEE Trans. Microwave Theory and Tech., Vol. 43, Mar. 1995, pp. 664-675.

[7] J. C. Goswami, A. K. Chan and C. K. Chui, "On solving first-kind integral equations using wavelets on a bounded interval," *IEEE Trans. Antennas Propagat.*, Vol. 43, Jun. 1995, pp. 614-622.

[8] G. Wang, "Application of wavelets on the Interval to the analysis of thin-wire antennas and scatterers," *IEEE Trans. Antennas Propagat.*, Vol. 45, May 1997, pp. 885-893.

[9] A. Cohen, I. Daubechie, and J-C Feauveau, "Bi-orthogonal bases of compactly supported wavelets", *Comm. Pure and Appl. Math.* 45 (1992), 485-560.

[10] W. Dahmen, A. Kunoth, and K. Urban, "Biorthogonal spline-wavelets on the interval --- stability and moment conditions", *Preprint*, 1996.