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Review Article

INVERSE SCATTERING IN ELECTROMAGNETIC AND ACOUSTIC TESTING OF MATERIALS

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Reconstruction of the property profile of a medium from measured data due to an illuminating wave is known as the inverse scattering problem. It arises in many fields of engineering and science, such as non-destructive evaluation, medical diagnosis, and seismology. The signifcance and challenge of the inverse scattering problems have attracted much research activity for decades, yet, it is far from being a resolved issue. Considerable effort is presently under way to find efficient and accurate inversion techniques. This is taking place independently in many disciplines.

The purpose of this review is to summarize the variety of methods in one dimension and in multiple dimensions. The sequence used to organize the review is from one dimensional problems to multiple dimension problems, from approximation methods to exact methods. Electromagnetic and acoustic methods are discussed in parallel.

1. Introduction

To introduce the basic equations an acoustic wave propagating in an elastic, isotropic and linear medium is considered first. If p, c, ρ and Δp, v, ρ' denote mean values of pressure, speed of sound, density and their disturbances associated with the propagation of the acoustic wave, the governing equations are

$$-\nabla p = \rho \boldsymbol{v}_t , \tag{1}$$

$$-\rho\nabla\cdot \boldsymbol{v} = \rho_t' , \tag{2}$$

$$p = c^2\rho' . \tag{3}$$

Substituting (3) into (2) for ρ', and taking the derivative of the result with respect to t, taking the divergence of (1), and eliminating $\rho\nabla\cdot\boldsymbol{v}_t$, one gets

$$\Delta p - p_{tt}/c^2 = u_1 , \tag{4}$$

where u_1 denotes the inhomogeneity

$$u_1 = -\nabla\rho\cdot\boldsymbol{v}_t = \nabla(\ln\rho)\cdot\nabla p . \tag{5}$$

In the last equation, (1) is used to remove $\boldsymbol{v}_t$.

In the electromagnetic wave case, the medium is assumed to be dielectric, nonmagnetic and dispersionless. The set of Maxwell's equations is

$$\nabla\times\boldsymbol{E} = -\boldsymbol{B}_t , \tag{6}$$

$$\nabla\times\boldsymbol{H} = \boldsymbol{D}_t , \tag{7}$$

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$$\nabla \cdot \boldsymbol{D} = 0 , \tag{8}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{9}$$

and the constitutive relations are

$$\boldsymbol{D} = \varepsilon \boldsymbol{E} , \tag{10}$$

$$\boldsymbol{B} = \mu \boldsymbol{H} . \tag{11}$$

Taking the curl of (6), and substituting (7) for $\nabla \times \boldsymbol{H}$, while neglecting $\nabla \mu$, the wave equation is obtained,

$$\Delta \boldsymbol{E} - \boldsymbol{E}_{tt}/c^2 = 0 , \tag{12}$$

where (8) and the constitutive relations are utilized. If one of the components of $\boldsymbol{E}$ is considered, (12) reduces to (4) with $u_1 = 0$. Thus, it suffices to study (13) in which f is either p or one component of $\boldsymbol{E}$ as needed:

$$\Delta f - f_{tt}/c^2 = u_2 , \tag{13}$$

where

$$\begin{aligned} &u_2 = \nabla(\ln \rho) \cdot \nabla f \quad \text{for an acoustic wave,} \\ &u_2 = 0 , \qquad\qquad\quad\;\; \text{for an electromagnetic wave} . \end{aligned} \tag{14}$$

Assuming time dependency $\exp(\mathrm{i}\omega t)$ (where ω is the angular frequency of the monochromatic wave) for f, (13) becomes a Helmholtz equation

$$(\Delta + k^2) f = u_2 , \tag{15}$$

where k is defined by

$$k^2 \triangleq \omega^2/c^2 \tag{16}$$

or

$$(\Delta + k_0^2 n) f = u_2 , \tag{17}$$

where k_0 is the wave number defined as

$$k_0^2 \triangleq \omega^2/c_0^2 \tag{18}$$

for the background medium with the speed of sound c_0, and where $n(x)$ is the index of refraction

$$n \triangleq c_0/c . \tag{19}$$

2. One-dimensional inverse scattering

In the one-dimensional case, an inhomogeneous layer of stratified elastic media extending from the origin to $x = 1$ in a homogeneous background medium is considered. A plane wave propagates along the x axis which coincides with the direction of stratification (see fig. 1). Then, eqs. (1)–(3) reduce to

$$-p_x = \rho v_t , \tag{20}$$

$$-\rho c^2 v_x = p_t , \tag{21}$$

where (3) is inserted into (2), and eqs. (8)–(9) reduce to (a TEM wave occurs)

$$E_x = -\mu H_t , \tag{22}$$

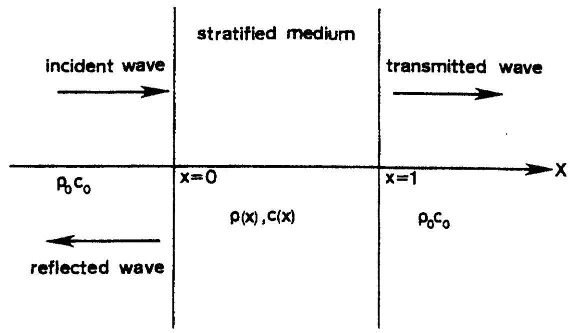


Fig. 1. Inverse scattering in one dimension.

$$H_x = \epsilon E_t \,. \tag{23}$$

By changing Δ to d_x^2 and dropping the inhomogeneous term u_2 on the left, (15) reduces to

$$f_{xx}(x) + k^2 f(x) = 0 \tag{24}$$

which will be treated by impediography.

On the other hand, an independent variable s is introduced below

$$s(x) = \int_0^x n(x)\,\mathrm{d}x \tag{25}$$

which is the travel distance in the background medium in the same time period as the wave travels from the origin to point x. Changing x to s, (17) is simplified to

$$f_{ss} + k_0^2 f = (\ln u_3)_s f_s \,, \tag{26}$$

where

$$u_3 = Z/Z_0 \quad \text{for an acoustical wave,}$$
$$u_3 = n^{-1} \quad \text{for an electromagnetic wave,} \tag{27}$$

and Z is the impedance of the medium:

$$Z = \rho c \,. \tag{28}$$

If a transform defined by

$$f = (u_3)^{1/2}\psi \tag{29}$$

is made, (26) further reduces to

$$\psi_{ss} + k_0^2\psi = q\psi \,, \tag{30}$$

where

$$q = [(\ln u_3)_s]^2/4 - (\ln u_3)_{ss}/2 \tag{31}$$

is a potential function which is compact and smooth if u_3 is smooth enough. Eq. (30) is the proper form with which the Distorted Wave Born Approximation and Gel'fand–Levitan's method will deal. To extract Z or c from q, (31) must be solved. We rewrite it into the form below

$$(u_3^{-1/2})_{ss} - q u_3^{-1/2} = 0 \tag{32}$$

which is a forward problem discussed in [1]. Also, the independent variable need to be changed back to x from s by inverting (25). This can be accomplished numerically.

In the time domain, the same approach can be applied to (13) and leads to

$$\psi_{\tau\tau} - \psi_{tt} = \gamma\psi_\tau , \tag{33}$$

where

$$\gamma = Z \quad \text{for an acoustical wave,}$$
$$\gamma = c \quad \text{for an electromagnetic wave,} \tag{34}$$

τ is the time for a wave traveling from origin to x, that is,

$$\tau = \int_0^x \mathrm{d}x/c(x) . \tag{35}$$

Eq. (33) is needed for the derivation of Balanis' method.

2.1. Impediography

Applying the well-known WKB approximation to one version of the one-dimensional wave equation (24) gives

$$f(x) = A_0 \exp\Big(-\mathrm{i} \int_0^x k(x)\,\mathrm{d}x\Big) + B_0 \exp\Big(\mathrm{i} \int_0^x k(x)\,\mathrm{d}x\Big) , \tag{36}$$

where A_0 and B_0 are arbitrary constants and may be determined from boundary conditions. This approximation suggests a solution in the following form

$$f(x) = A(x) \exp\Big(-\mathrm{i} \int_0^x k(x)\,\mathrm{d}x\Big) + B(x) \exp\Big(\mathrm{i} \int_0^x k(x)\,\mathrm{d}x\Big) , \tag{37}$$

such that A_0 and B_0 are respectively the first terms of the power series expansions of $A(x)$ and $B(x)$ with respect to a small parameter.

By substituting this solution back into (20, 21) or (22, 23), and neglecting high order reflections, $A(x)$ and $B(x)$ satisfy

$$\mathrm{d}A/\mathrm{d}x = -\frac{B}{2}\,\frac{\mathrm{d}(\ln \zeta)}{\mathrm{d}x}\, \exp\Big(2\mathrm{i} \int_0^x k\,\mathrm{d}x\Big) , \tag{38}$$

$$\mathrm{d}B/\mathrm{d}x = -\frac{A}{2}\,\frac{\mathrm{d}(\ln \zeta)}{\mathrm{d}x}\, \exp\Big(-2\mathrm{i} \int_0^x k\,\mathrm{d}x\Big) , \tag{39}$$

where $\zeta(x) = Z = \rho c$ is the impedance in the acoustical case, and $\zeta(x) = c$ is the local speed of wave in the electromagnetic case. Now, the reflection coefficient may be approximated as

$$R(\omega) = B(0)/A(0) \approx -\frac{1}{2} \int_\infty^0 \exp\Big(-2i \int_{x_0}^x k\,\mathrm{d}x\Big)\, \mathrm{d}(\ln \zeta) , \tag{40}$$

where, again, only the first-order reflection is taken into account. To simplify (40), a new independent variable is introduced as the two-way traveling time from 0 to x, that is,

$$\tau(x) = 2 \int_0^x \mathrm{d}x / c(x) . \tag{41}$$

Inserting (41) and the relation $k = \omega / c$, (40) can be rewritten as

$$R(\omega) = \frac{1}{2} \int_0^\infty \mathrm{d}(\ln \zeta) \exp(-\mathrm{i}\omega\tau) . \tag{42}$$

The impulse response is related to the reflection coefficient inverse Fourier transform as

$$R(\omega) = \int_0^\infty r(\tau) \exp(-\mathrm{i}\omega\tau) \, \mathrm{d}\tau . \tag{43}$$

Comparing the last two equations, one concludes that

$$r(\tau) = \frac{1}{2} (\ln \zeta(\tau))_\tau . \tag{44}$$

This simple equation is known as the impediography equation which relates the ζ directly with the impulse response. Integrating (44) over $(0, \tau)$, another form is obtained

$$\int_0^\tau r(t) \, \mathrm{d}t = \frac{1}{2} \ln(\zeta(\tau) / \zeta(0)) . \tag{45}$$

At this stage, ζ is a function of independent variable τ. It is necessary to cast τ back to x by inversion of (41)

$$x = \frac{1}{2} \int_0^\tau c(\tau) \, \mathrm{d}\tau . \tag{46}$$

Thus, the impedance (or the local speed of wave) can be calculated via (44) (or (45)) and (46) given the impulse response and the impedance (or speed) at $x \leqslant 0$.

Impediography was extensively discussed in the literature during the early 70's [2, 3]. The impediography equation has been derived either from the wave equation [4] or from discrete reflection analysis [5]. The above derivation makes use of the WKB approximation [6]. Since the WKB approximation is a first order approximation to the solution of a wave equation, impediography is valid only for media with sufficiently small spatial variation, so that the reflection is an order of magnitude, or so, smaller than the transmission [4]. See [7, 8] for examples.

2.2. *The Distorted Wave Born Approximation* (*DWBA*)

DWBA has evolved from the well-known two-potential formula which was derived in context of particle scattering in [9]. In the following, the DWBA is derived from the wave equation. Assuming that q_0 is chosen to be the reference potential to equation (30) and, moreover, ψ_0 is the corresponding solution, that is,

$$(\psi_0)_{ss} + k_0^2 \psi_0 = q_0 \psi_0 . \tag{47}$$

Multiplying (30) by ψ_0, (47) by ψ, and subtracting the products, yields

$$\psi_0\psi_{ss} - \psi(\psi_0)_{ss} = (q - q_0)\psi\psi_0 . \tag{48}$$

For ψ and ψ_0, there exist the following asymptotic forms

$$\psi = \exp(-ik_0 s) + R\exp(ik_0 s) \qquad s \leq 0 , \tag{49}$$

$$\psi = T\exp(-ik_0 s) \qquad s \to \infty , \tag{50}$$

$$\psi_0 = \exp(-ik_0 s) + R_0\exp(ik_0 s) \qquad s \leq 0 , \tag{51}$$

$$\psi_0 = T_0\exp(-ik_0 s) \qquad s \to \infty . \tag{52}$$

Integrating over $(0, \infty)$, (48) becomes

$$\int_0^\infty (\psi_0\psi_{ss} - \psi(\psi_0)_{ss})\,ds = \int_0^\infty \psi(q - q_0)\psi_0\,ds . \tag{53}$$

Integrating by parts, the left hand side of the equation can be evaluated as

$$[\psi_0\psi_s - \psi(\psi_0)_s]_0^\infty = i2k_0(R_0 - R) , \tag{54}$$

where (49)–(52) are utilized. Substituting this result into (53) yields the two potential formula:

$$2k_0(R - R_0)/i = \int_0^\infty \psi(q - q_0)\psi_0\,ds . \tag{55}$$

Now, derivation of the DWBA can commence. Assuming that q_0 can be chosen such that it is very close to q, then, ψ is expected to be close to ψ_0 too, and ψ_0 can be used as ψ in (55). If so, (55) reduces to

$$2k_0(R - R_0)/i = \int_0^\infty (q - q_0)(\psi_0)^2\,ds , \tag{56}$$

which is the DWBA. In particular, it reduces to the Born approximation

$$2k_0 R/i = \int_0^\infty q\exp(-i2k_0 s)\,ds , \tag{57}$$

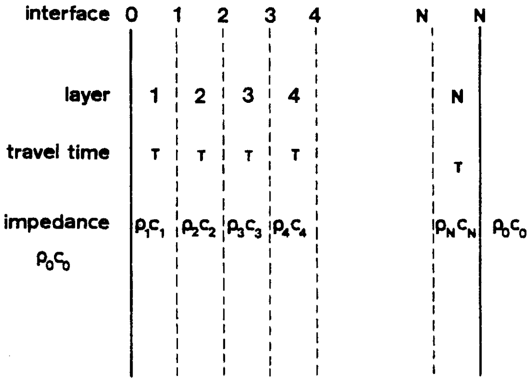
if q_0 vanishes, and therefore $\psi_0 = \exp(-ik_0 x)$. An equation similar to the impediography equation can be derived from this.

If the frequency for each experiment is varied, N projections of $q - q_0$ onto ψ_0^2 are obtained from N experiments. Then, q can be recovered from the projections by methods of, for example, Gram–Schmit orthogonalization and singular value decomposition [10, 11]. They also suggest that the DWBA can be applied repeatedly to new values of q to achieve higher precision. However, the constrains for the convergence of the iterative DWBA method are unknown to the authors and need further study.

2.3. Goupillaud's method

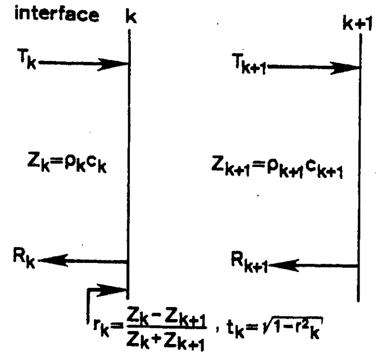
Goupillaud's method [12, 13] is based on discrete analysis of wave scattering in a layered medium. The analysis relies heavily upon physical intuition, in particular, the decomposition of transmitted and reflected waves, while other methods employ more mathematical arguments. Consider the well-known Goupillaud layered medium, i.e., the travel time across each layer is the same, say, τ. (See fig. 2(a)).

To derive the method, the transmission and reflection coefficients at the kth interface between the kth medium and the $(k+1)$th medium are assumed to be t_k and r_k respectively. In the setting of time



(a)

Fig. 2 (a). Goupillaud layered medium.



(b)

Fig. 2 (b). Derivation of M matrix.

domain Fourier transform, the transmitted and reflected fields for that sections of the medium, shown in fig. 2(b), can be expressed in matrix form, by noting that the forward and backward transmission coefficients are both $(1-r_k^2)^{1/2}$ and the backward reflection coefficient is $-r_k$,

$$\begin{bmatrix} T_k \\ R_k \end{bmatrix} = M_k \begin{bmatrix} T_{k+1} \\ R_{k+1} \end{bmatrix}, \tag{58}$$

where

$$M_k = t_k^{-1} \begin{bmatrix} W & r_k W^{-1} \\ r_k W & W^{-1} \end{bmatrix}, \qquad W = \mathrm{e}^{\mathrm{i}\omega\tau} \tag{59}$$

and T_k and R_k denote the forward and backward components of the total field in the kth medium. Employing (58, 59), a system with N layers can be described. Eq. (58) is used $N+1$ times for $N+1$ interfaces to get

$$\begin{bmatrix} T_0^N \\ R_0^N \end{bmatrix} = M_0 M_1 \cdots M_N \begin{bmatrix} T_{N+1} \\ 0 \end{bmatrix}. \tag{60}$$

Let a matrix M^k be defined by

$$M^k = M_0 M_1 \cdots M_k = (t_0 t_1 \cdots t_k)^{-1} \left\| \begin{bmatrix} W^{k+1}F(k, W^{-1}) & W^{-k-1}G(k, W) \\ W^{k+1}G(k, W^{-1}) & W^{-k-1}F(k, W) \end{bmatrix}, \tag{61}$$

where F and G are given by the following recursive formulas:

$$F(k, W) = F(k-1, W) + r_k W^{2k} G(k-1, W^{-1}), \tag{62}$$

$$G(k, W) = G(k-1, W) + r_k W^{2k} F(k-1, W^{-1}), \tag{63}$$

and initially,

$$F(0, W) = 1, \qquad G(0, W) = r_0. \tag{64}$$

By repeated use of (62, 63), F and G are built from (64); The form of F is:

$$F(k, W^{-1}) = 1 + F_k(1)W^{-2} + F_k(2)W^{-4} + \cdots + F_k(k)W^{-2k}. \tag{65}$$

G has the same form as F. Since $\det(M_k) = t_k$, and $\det(M^k) = \det(M_0 M_1 \cdots M_k) = \det(M_0)\det(M_1)\cdots \det(M_k)$, F and G have the relation below

$$F(k, W)F(k, W^{-1}) - G(k, W)G(k, W^{-1}) = \prod_{i=0}^{k} t_i^2 . \tag{66}$$

R_0^N can be derived from (60) in terms of F and G as

$$R_0^N = T_0^N \cdot G(N, W^{-1})/F(N, W^{-1}) . \tag{67}$$

In particular, the reflection coefficient R of the system is (67) if one sets $T_0^N = 1$, that is, the incident wave is set to unity,

$$R = G(N, W^{-1})/F(N, W^{-1}) . \tag{68}$$

If all the coefficients of all interfaces are known, one can compute F and G by (62)–(64), and therefore, can calculate the reflection coefficient by (68).

In the context of inverse problems, the r's need to be related to R. Let the R be expressed as a power series of W:

$$R_0^N = d_0 + d_1 W^{-2} + d_2 W^{-4} + \cdots . \tag{69}$$

Then, the d's of R_0^N must agree with the d's of $R_0^k (k \leq N)$ up to and including the $-2k$ power of W since the first order reflection from the kth interface takes $2k\tau$ time units. This permits us to accumulate F and G while calculating r's. To see that, (62, 63) are inserted into (68):

$$R = \frac{G(N-1, W^{-1}) + r_N W^{-2N} F(N-1, W)}{F(N-1, W^{-1}) + r_N W^{-2N} G(N-1, W)} , \tag{70}$$

or

$$RF(N-1, W^{-1}) + r_N W^{-2N} RG(N-1, W) = G(N-1, W^{-1}) + r_N W^{-2N} F(N-1, W) . \tag{71}$$

Applying (70) again at the second R in (71) results in

$$RF(N-1, W^{-1}) = G(N-1, W^{-1}) + \prod_{i=0}^{N-1} t_i^2 r_N W^{-2N} / F(N, W^{-1}) , \tag{72}$$

where (66) and (62) were also used. Inverting $F(N, W^{-1})$, the following is obtained

$$RF(N-1, W^{-1}) = G(N-1, W^{-1}) + \prod_{i=0}^{N-1} r_N W^{-2N} + \cdots . \tag{73}$$

Equating the coefficients of W^{-2N} of (73) gives us the desired formulas

$$r_k = \frac{\sum_{i=1}^{k} d_i F_{k-1}(k-i)}{\prod_{i=0}^{k-1} (1 - r_i^2)} \quad k \geq 1 , \tag{74}$$

with $r_0 = d_0$.

Equipped with (74), the inversion can now be performed. First, set $r_0 = d_0$, $F(0, W^{-1}) = 1$ and $G(0, W^{-1}) = r_0$. Then, apply (74) to F, G, r_0 and d_1 to calculate r_1; apply (62, 63) to calculate $F(1, W^{-1})$ and $G(1, W^{-1})$. Again, (74) is ready to use for r_2. In this way, all reflection coefficients needed can be found. Finally, the relation below permits the reconstruction of the impedance $Z(k)$ if any one of them, for instance, z_0, is known.

$$\prod_{i=0}^{k} (1 - r_i)/(1 + r_i) = Z(k + 1)/z_0 . \tag{75}$$

Goupillaud's method is an exact method in the discrete case. For a continuous medium, it gives better approximation for finer slabs. Since it incorporates both first and high order reflections, Goupillaud's method is still valid in case of large impedance variations. See [5, 14] for the application of Goupillaud's method, and comparison with other methods.

2.4. The methods of Gel'fand–Levitan and Marchenko

Gel'fand and Levitan (GL) derived an elegant method to reconstruct the second order equation from the given spectrum [15]. More precisely, in our case, consider (30) with a boundary condition

$$\psi_s(0, k_0) - h\psi(0, k_0) = 0 , \tag{76}$$

assuming q a sufficiently smooth function on any finite interval. Their work was devoted to the solution of the following problems: determine the existence of an equation of the form (30) provided the spectral function $\rho(k_0)$ is known; and reconstruct the function q. The first treatment of this type of problem is given in [16]. The uniqueness and solvability of the inverse problem was established by Marchenko and Krein in [17] and [18] respectively. An outline of the derivation given by Gel'fand and Levitan follows.

Assuming the spectral function of (30) is known, namely,

$$\rho(k_0) = \begin{cases} (2/\pi\sqrt{k_0} + \sigma(k_0) & k_0 > 0 , \\ \sigma(k_0) & k_0 < 0 . \end{cases} \tag{77}$$

Then, the equation possesses the eigenfunction expressed in the form

$$\phi(x, k_0) = \cos\sqrt{k_0 x} + \int_0^x K(x, t) \cos\sqrt{k_0 t}\, dt . \tag{78}$$

By an orthonormal process, that is let $\phi(x, k_0)$ satisfy

$$\int_{-\infty}^{\infty} \phi(x, k_0)\phi(y, k_0)\, d\rho(k_0) = \delta(x - y) , \tag{79}$$

or

$$\int_{-\infty}^{\infty} \phi(x, k_0) \cos\sqrt{k_0 y}\, d\rho(k_0) = 0 \quad \text{for } y < x , \tag{80}$$

since $\phi(y, k_0)$ is a combination of $\cos\sqrt{k_0 y}$, which is implied by (78). We obtain the GL equation, which is uniquely solvable,

$$f(x, y) + K(x, y) + \int_0^x f(y, t)K(x, t)\, dt = 0 , \tag{81}$$

where

$$K_x(x, x) = q(x)/2 \tag{82}$$

and

$$f(x, y) = \int_{-\infty}^{\infty} \cos\sqrt{k_0 x}\cos\sqrt{k_0 y}\, \mathrm{d}\sigma(k_0)\,. \tag{83}$$

If the spectral function $\rho(k_0)$ is known, the above GL theory solves the inverse spectral problem. However, what is commonly known is the system response, such as the impulse response, rather than the spectral response. A more direct approach was taken in [19] which establishes the following: (Marchenko's equation)

$$K(x, y) = -r(x + y) - \int_{-t}^{x} K(x, t) r(y + t)\, \mathrm{d}t\,, \tag{84}$$

where $r(t)$ is the impulse response measured at origin.

Now, the inversion is in order. Solve (84) for $K(x, y)$ from the given data $r(t)$. Then, (82) can be used to obtain q. See [20–23] for examples which are based on this method. The GL theory was derived in a restricted sense in the time domain in [24, 25].

2.5. Balanis' method

As in [26], consider equation (33). The impulse response of the system can be represented as follows [27]; if γ is continuous and vanishes for $\tau \leqslant 0$ and $\tau \to \infty$,

$$\psi(\tau, t) = \delta(\tau - t) + r(\tau + t)\,, \quad \tau \leqslant 0\,, \tag{85}$$

$$\psi(\tau, t) = \delta(\tau - t) + r(\tau + t) - K_t(\tau, t) + \int_{-\tau}^{\tau} K(\tau, x) r'(t + x)\, \mathrm{d}x\,, \tag{86}$$

where $r'(t)$ stands for the derivative of r with respect to t, and $K(\tau, t)$ satisfies

$$K_{\tau\tau} - K_{tt} - \gamma K_\tau = 0\,, \tag{87}$$

$$K(\tau, t) = 0, \quad t \leqslant -\tau\,, \quad t > \tau \tag{88}$$

and

$$2K_\tau(\tau, \tau) - \gamma K(\tau, \tau) = \gamma\,, \tag{89}$$

Integrating (86) over the range $(-\tau, t)$ with respect to t

$$\int_{-\tau}^{t} r(\tau + t)\, \mathrm{d}t - \int_{-\tau}^{t} K_t(\tau, t)\, \mathrm{d}t + \int_{-\tau}^{t}\int_{-\tau}^{\tau} K(\tau, x) r'(t + x)\, \mathrm{d}x\, \mathrm{d}t = 0\,, \tag{90}$$

with $\psi(\tau, t) = 0$ and $\delta(\tau - t) = 0$ for $t < \tau$. Reversing the order of integration of the last term, and noting that $K(\tau, -\tau) = 0$ and $r(t < 0) = 0$, one finds that

$$\int_{-\tau}^{t} r(\tau + t)\, \mathrm{d}t - K(\tau, t) + \int_{-\tau}^{\tau} K(\tau, x) r(t + x)\, \mathrm{d}x = 0\,. \tag{91}$$

Thus, the following integral equation (Balanis' equation) is obtained:

$$\int_{0}^{t+\tau} r(t)\,\mathrm{d}t - K(\tau, t) + \int_{-t}^{\tau} K(\tau, x) r(t+x)\,\mathrm{d}x = 0\,, \quad |t| < \tau\,. \tag{92}$$

This integral equation has a unique solution for $K(\tau, t)$ provided $0 \leq |R(\omega)| < 1$, where

$$R(\omega) = \int_{-\infty}^{\infty} r(t)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t\,. \tag{93}$$

Now, the process of inversion is clear. It contains two steps. First, (92) is solved for $K(\tau, t)$ from the impulse response $r(t)$. Then, $\gamma(\tau)$ can be readily recovered by means of (89) which can be written, explicitly for the potential function, as

$$\gamma(\tau) = 2K_\tau(\tau, \tau)/[1 + K(\tau, \tau)] \tag{94}$$

Examples of Balanis' method can be found in refs. [28–33]. It was reported that, comparing with other exact methods, the method provides an efficient numerical implementation with less storage requirement and appears to be robust with respect to noise data.

2.6. Method of characteristics

The method of characteristics was first developed by Courant et al. [34] for the solution of nonlinear hyperbolic differential equations. They also proved the stability of the procedure. The idea was introduced to inverse problems by Santosa and Schwetlick [35]. See [36, 37] for other examples. Starting from scratch, by inserting (21) and $v = u_t$ (where u is the displacement), (20) reads

$$(\rho c^2 u_x)_x = \rho u_{tt}\,. \tag{95}$$

Changing the independent variable from x to τ defined by (35) in (95), Webster's horn equation is obtained

$$Z u_{tt} = (Z u_\tau)_\tau\,. \tag{96}$$

Now, following the standard procedure, the first step is to rewrite the second order differential system (96) in the form of a first order hyperbolic system. By setting $v_1 = u_t$, and $v_2 = Z u_\tau$, (96) becomes

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_t = \begin{bmatrix} 0 & Z^{-1} \\ Z & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_\tau, \tag{97}$$

or in matrix form

$$\boldsymbol{V}_t = \boldsymbol{A} \boldsymbol{V}_\tau\,. \tag{98}$$

The next step is to diagonalize the matrix $\boldsymbol{A}$, so that

$$\boldsymbol{T}\boldsymbol{V}_t = -\boldsymbol{\Lambda}\boldsymbol{T}\boldsymbol{V}_\tau\,, \tag{99}$$

where $\boldsymbol{T}$ is the transform matrix, $\Lambda = -\boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{-1} = \mathrm{diag}(\lambda_1, \lambda_2)$. It is easy to show that

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{100}$$

if

$$\boldsymbol{T} = \begin{bmatrix} -Z & 1 \\ Z & 1 \end{bmatrix}. \tag{101}$$

Now, it is clear that the characteristic directions are

$$\mathrm{d}t/\mathrm{d}\tau = \lambda_i \quad \text{for} \quad i = 1, 2\,, \tag{102}$$

and the characteristic derivative is

$$\mathrm{d}V/\mathrm{d}\tau = \partial V/\partial \tau + (\partial V/\partial t)(\mathrm{d}t/\mathrm{d}\tau) = \partial V/\partial \tau + \lambda_i\, \partial V/\partial t \quad \text{for} \quad i = 1, 2\,. \tag{103}$$

Finally, (101)–(103) are inserted into (99) to obtain

$$Z(\mathrm{d}v_1/\mathrm{d}\tau) = \mathrm{d}v_2/\mathrm{d}\tau \quad \text{for} \quad \mathrm{d}t/\mathrm{d}\tau = \lambda_1 = 1\,, \tag{104}$$

and

$$-Z(\mathrm{d}v_1/\mathrm{d}\tau) = \mathrm{d}v_2/\mathrm{d}\tau \quad \text{for} \quad \mathrm{d}t/d\tau = \lambda_2 = -1\,. \tag{105}$$

It is convenient to view the inverse problem in the t–τ plane as a boundary value problem (illustrated in fig. 3). Commonly given data are the exciting pressure $f(t)$ and the measured displacement $g(t)$ at $x = 0$, that is,

$$\rho c^2 u_x(0, t) = f(t) \quad \text{and} \tag{106}$$

$$u(0, t) = g(t)\,. \tag{107}$$

Utilizing these boundary conditions, v_1 and v_2 are expressed in terms of f and g as follows

$$v_1(0, t) = u_t(0, t) = g'(t)\,, \tag{108}$$

$$v_2(0, t) = Zu_\tau(0, t) = f(t)\,, \tag{109}$$

where (34) is used. By realizing the causality of a physical system, another pair of boundary conditions is obtained:

$$v_1 = v_2 = 0 \quad \text{for} \quad \tau \geqslant t \geqslant 0 \quad \text{and} \quad t < 0\,. \tag{110}$$

Employing the governing equations (104, 105) in the domain $t > \tau > 0$, together with the boundary conditions (108, 109), one should find $Z(t > \tau > 0)$. For example, [35] employs the finite difference method to discretize the equation pair and to calculate the impedance profile $Z(\tau)$ at discrete points. [36] uses a local regulation method to improve the error caused by the finite difference approximation.

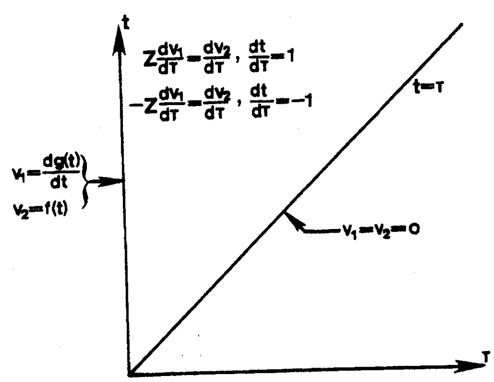


Fig. 3. A boundary problem.

3. Multiple dimension inverse scattering

The system considered is illustrated in fig. 4. There is a finite inhomogeneous object embedded in a homogeneous medium. A probing wave illuminates the object. Thus, the scattered wave around it can be measured. It is hoped that the property profile of the object can be reconstructed by processing the measurements.

The idea used in the one-dimensional case may be extended to convert (17) into a desired Schrödinger form. First, the coordinate variable x is changed to r as defined by

$$r_i = \int_{x_0 i}^{xi} n(x)\,\mathrm{d}x_i\,. \tag{111}$$

By the above coordinate change, (17) becomes

$$\Delta f + k_0^2 f = \nabla(\ln u_4)\cdot\nabla f\,, \tag{112}$$

where

$$u_4 = Z/Z_0 \quad \text{in the acoustical case,} \quad \text{and}$$

$$u_4 = n^{-1} \quad \text{in the electromagentic case.} \tag{113}$$

Then, a transform is given as

$$f = u_4^{1/2}\psi \tag{114}$$

which leads (112) to

$$\Delta\psi + k_0^2\psi = -O\psi\,, \tag{115}$$

where the object function O is defined as

$$O = -\nabla(\ln u_4)\cdot\nabla(\ln u_4)/4 + \Delta(\ln u_4)/2\,. \tag{116}$$

To solve eq. (115), a Green's function is introduced by setting it to satisfy

$$(\Delta + k_0^2)g(r) = -\delta(r)\,. \tag{117}$$

The Green's function represents the field of a point source at origin and can be derived as

$$g(r) = \exp(\mathrm{i}k_0R)/(4\pi R)\,, \tag{118}$$

where $R = |r|$. Since the system considered is linear, the solution of (115) may be written in terms of the Green's function by superposition:

$$\psi(r) = \psi_0 + \int_v g(r - r')O(r')\psi(r')\,\mathrm{d}r'\,, \tag{119}$$

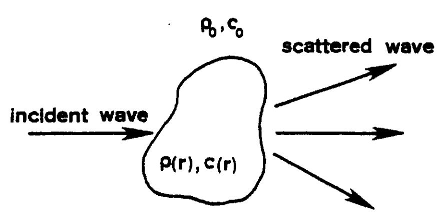


Fig. 4. A general model.

where ψ_0 is the incident wave, that is, it satisfies

$$\Delta\psi_0 + k_0^2\psi_0 = 0 \tag{120}$$

and v the volume occupied by the inhomogeneous body.

Earlier research work assumes that waves propagate in straight-paths. Under this approximation, a class of geometrical optics-based methods has been developed. The methods which reconstruct property profiles from projections fall into two categories: spatial and Fourier domain reconstruction.

An equation can be established from the accumulation of losses (to reconstruct an attenuation profile) or traveling time (to reconstruct a speed profile) for a ray. For a set of rays, a set of equations is established. To solve the equations, conventional direct methods give unacceptable noisy solutions [38]. On the other hand, the projection method [39], the method of singular-value decomposition [40] and the algebraic reconstruction technique (ART) [41–43] achieve various accuracies. ART is among the most efficient methods.

The set of equations can be replaced by a more accurate integral equation which leads to the convolution method [44, 45] and the Fourier transform method [46, 47]. The former, like the methods mentioned in the last paragraph, is a spatial domain method. The latter is the only method which works in the Fourier domain according to the projection-slice theorem. These two methods, specially the Fourier transform method, are very efficient computationally.

Despite the fact that some of the above methods are currently in practical use, straight-path is obviously a restrictive assumption for acoustical and electromagnetic waves. Some suggestions, for example, ray-tracing, have been proposed to accommodate corrections. However, for improved accuracy, diffraction must be taken into account.

3.1. The Born and Rytov approximations

These methods are the first attempts to take the effects of diffraction and refraction in weakly inhomogeneous media into account. The Born approximation is simpler. Let the total field $\psi(\boldsymbol{r})$ expressed as a sum of the incident field $\psi_0(\boldsymbol{r})$ and the scattered field $\psi_s(\boldsymbol{r})$. Then, (119) can be rewritten as

$$\psi_s = \psi - \psi_0 = \int gO\psi_0 \, d\boldsymbol{r}' + \int gO\psi_s \, d\boldsymbol{r}' . \tag{121}$$

If the scattered component is smaller than the incident field and may be neglected, the following approximation is reasonable:

$$\hat{\psi}_s(\boldsymbol{r}) = \int g(\boldsymbol{r} - \boldsymbol{r}')O(\boldsymbol{r}')\psi_0(\boldsymbol{r}') \, d\boldsymbol{r}' . \tag{122}$$

This is the so-called first-order Born approximation. It is easy to see that $\psi_0 + \psi_s$ is a better approximation of ψ than ψ_0 alone. Thus, the equation

$$\hat{\psi}_s^{(2)}(\boldsymbol{r}) = \int g(\boldsymbol{r} - \boldsymbol{r}')O(\boldsymbol{r}')[\psi_0(\boldsymbol{r}') + \hat{\psi}_s(\boldsymbol{r}')] \, d\boldsymbol{r}' , \tag{123}$$

results in a more accurate solution. $\hat{\psi}_2^{(2)}$ is the second-order Born approximation. In this way, the nth-order Born approximation can be obtained:

$$\hat{\psi}_s^{(n)}(\boldsymbol{r}) = \int g(\boldsymbol{r} - \boldsymbol{r}')O(\boldsymbol{r}')[\psi_0(\boldsymbol{r}') + \hat{\psi}_s^{(n-1)}(\boldsymbol{r}')] \, d\boldsymbol{r}' . \tag{124}$$

To derive the Rytov approximation, a transformation is necessary. ϕ is defined as the complex phase and set to

$$\psi(\boldsymbol{r}) = \exp[\phi(\boldsymbol{r})] \,. \tag{125}$$

This is substituted into (115) and leads to a Riccati equation:

$$\Delta\phi + (\nabla\phi)^2 + k_0^2 = -O \,. \tag{126}$$

Let

$$\psi_0 = \exp(\phi_0) \tag{127}$$

and

$$\phi = \phi_0 + \phi_s \,. \tag{128}$$

Substitute (127) into (120) to yield

$$\Delta\phi_0 + \nabla\phi_0 \cdot \nabla\phi_0 + k_0^2 = 0 \,. \tag{129}$$

Substituting (128) into (126) gives

$$\Delta\phi_s + 2\nabla\phi_0 \cdot \nabla\phi_s + \nabla\phi_s \cdot \nabla\phi_s = -O \,, \tag{130}$$

where (129) is used. With a monochromatic plane incident wave, $\psi_0 = \exp(-\mathrm{i}\boldsymbol{k}_0 \cdot \boldsymbol{r})$, and $\phi_0 = -\mathrm{i}\boldsymbol{k}_0 \cdot \boldsymbol{r}$. Thus;

$$\Delta(\psi_0\phi_s) = \psi_0(\Delta\phi_s + 2\nabla\phi_0 \cdot \nabla\phi_s - k_0^2\phi_s) \,. \tag{131}$$

Arranging the above identity results in:

$$(\Delta + k_0^2)(\psi_0\phi_s) = \psi_0(\Delta\phi_s + 2\nabla\phi_0 \cdot \nabla\phi_s) \,. \tag{132}$$

Combining (130) and (132) yields,

$$(\Delta + k_0^2)(\psi_0\phi_s) = -(O + \nabla\phi_s \cdot \nabla f_s)\psi_0 \,, \tag{133}$$

which is equivalent to the following integral equation

$$\psi_0\phi_s(\boldsymbol{r}) = \int g(\boldsymbol{r} - \boldsymbol{r}')(O + \nabla\phi_s \cdot \nabla\phi_s)(\boldsymbol{r}')\psi_0(\boldsymbol{r}')\,\mathrm{d}\boldsymbol{r}' \,. \tag{134}$$

Provided $\nabla\phi_s \cdot \nabla\phi_s$ is negligible compared to O, this equation reduces to the Rytov approximation:

$$\hat{\phi}_s(\boldsymbol{r}) = (\psi_0)^{-1} \int g(\boldsymbol{r} - \boldsymbol{r}')O(\boldsymbol{r}')\psi_0(\boldsymbol{r}')\,\mathrm{d}\boldsymbol{r}' \,. \tag{135}$$

The Born and Rytov approximations are equivalent for weak scattering. To see that, substitute (132) into (134) to reach

$$\hat{\phi}_s = \hat{\psi}_s/\psi_0 \,, \tag{136}$$

or

$$\hat{\psi}_s^{(B)} = \psi_0 \ln(\hat{\psi}^{(R)}/\psi_0) = \hat{\psi}_s^{(R)}[1 - (\hat{\psi}_s^{(R)}/\psi_0)/2 + \cdots] \,, \tag{137}$$

since $\hat{\psi}^{(R)} = \psi_0 \exp(\hat{\psi}_s) = \psi_0 + \hat{\psi}_s^{(R)}$. This shows that the Born approximation is the first term of the series expansion of the Rytov approximation. However, their validities are quite different [48]. Under the Born approximation, the change in phase between the incident wave and the total wave must be less than π, so that the product of the size and the refractive index of an ojbect is vital. On the other hand, only the change in scattered phase over one wavelength has great importance to the Rytov approximation. Moreover, a recent paper [49] indicates that the Rytov approximation error is the third-order perturbation term in contrast to the first-order of the Born approximation error. Thus, the Rytov approximation is generally superior [50, 51].

Direct computation of the approximations from (122) and (134) is usually expensive and inefficient. Taking the Fourier transform of both leads to the Fourier Diffraction Projection Theorem provided the insonification wave is a monochromatic plane wave [52]. The theorem claims that the Fourier transform of the field data along a line is proportional to the Fourier transform of the object function along a circular arc. Thus, multidimensional FFT will make the computation very efficient.

There are a variety of methods to implement the idea. The experimental procedures, for example, rotating either the object or the transmitter [53], varying the frequency of the incident wave [54] and synthesizing an aperture [55], were proposed to fill up the Fourier space of an object function. To measure the data, either transmission [56] or reflection [57] modes can be adopted. Furthermore, there are three distinct algorithms proposed to convert a non-uniform grid to a uniform grid for use of FFT: filtered backpropagation [56], interpolation [52] and interpolation-free [55]. Their computational complexity are of the order of $O(N^4)$, $O(N^2 \log N)$ and $O(N^3)$ respectively. With the Fourier Diffraction Projection Theorem, it is possible to implement high order Born or Rytov approximations.

3.2. Inverse moment methods

Moment methods are widely used for forward scattering problems. The application to inverse scattering is formally introduced in [58]. A particular moment method called sinc basis, multiple source, moment method was proposed to permit strong scattering.

Starting with (119), a set of base functions e_j is chosen such that the function $(O\psi)(\boldsymbol{r})$ can be expanded as

$$(O\psi)(\boldsymbol{r}) = \sum_j a_j e_j(\boldsymbol{r}) . \tag{138}$$

Substituting (138) into (119) and switching the order of integration and summation yields

$$\psi(\boldsymbol{r}) = \psi_0(\boldsymbol{r}) + \sum_j a_j \int e_j(\boldsymbol{r}')g(\boldsymbol{r} - \boldsymbol{r}')\, \mathrm{d}\boldsymbol{r}' . \tag{139}$$

Since e_j and g are known functions, the integration can be performed directly. Set

$$q_j(\boldsymbol{r}) = \int e_j(\boldsymbol{r}')g(\boldsymbol{r} - \boldsymbol{r}')\, \mathrm{d}\boldsymbol{r}' \tag{140}$$

and now (139) can be rewritten as

$$\psi(\boldsymbol{r}) = \psi_0(\boldsymbol{r}) + \sum_j a_j q_j(\boldsymbol{r}) . \tag{141}$$

(138) and (141) may be coupled to solve for $O(\boldsymbol{r})$ numerically provided $\psi(\boldsymbol{r})$ is given over enough points.

The sinc basis, multiple source, moment method is a realization of the above procedure. In the method, the base functions are set to be the shifted version of multiple dimensional sinc functions [59]:

$$e_j(\boldsymbol{r}) = \prod_k \mathrm{sinc}[(\pi/h)(r_k - n_{kj}h)] , \tag{142}$$

where $\boldsymbol{r} = (r_1, \ldots, r_K)$; h the step size between grid points in each coordinate; j an index number specifying individual grid points at $(n_{lj}h, \ldots, n_{Kj}h)$; n's integers. Then, whenever $O\psi$ is a band limited function,

$$a_j = (O\psi)(\boldsymbol{r}_j) \tag{143}$$

and

$$b_{lj} = q_j(\boldsymbol{r}_l) = \int e_j(\boldsymbol{r}')g(\boldsymbol{r}_l - \boldsymbol{r}')\,\mathrm{d}\boldsymbol{r}'[60]\,. \tag{144}$$

Now, (141) reads

$$\psi_l = \psi_{0l} + \sum_j a_j b_{lj} \tag{145}$$

or in matrix form,

$$[\psi] = [\psi_0] + [B][A]\,, \tag{146}$$

where l is an index to one of the points of interest. To improve the condition of the ill-posed problem, multiple sources were suggested. In this case, (145) becomes, for the field outside the object,

$$\psi_{lm} = f_{0lm} + \sum_j a_{jm} b_{lj}\,, \qquad l = 1, \ldots, L; \quad m = 1, \ldots, M; \quad j = 1, \ldots, N\,; \tag{147}$$

where L is the number of receivers used in each experiment, M the number of experiments with different views, N the number of grid points of the image. (147) is still a valid expression for the field inside the object. However, the coefficients, b_{lj}, are different in this case. To show the difference, a letter c replaces b in (147) as

$$f_{lm} = f_{0lm} + \sum_j a_{jm} c_{lj}\,, \qquad l = 1, \ldots, L; \quad m = 1, \ldots, M; \quad j = 1, \ldots, N\,. \tag{148}$$

The matrix form is the same as (146) except that $[\psi]$, $[\psi_0]$ and $[A]$ are also matrices now.

To solve the nonlinear system in (147, 148) for $O(4r)$ and $\psi(r)$, four procedures were suggested in [58], and the second procedure, so called the alternating variable linear Kaczmarz method, was further studied in [61]. With the help of the FFT and backprojection, the total computation complexity of the procedure reduces to $O(N^3 \log N)$ for an $N \times N$ image [62]. Unfortunately, the iterative solution behaves as the Born approximation does, that is, the method fails to converge to the solution whenever the phase difference between the incident wave and the total wave is greater than π [63]. A direct method may improve the solution considerably.

3.3. *The methods of Gel'fand–Levitan and Marchenko*

The Gel'fand–Levitan and Marchenko's inversion theory were extended to three dimensions by Newton [64–66]. The derivation of these theories is complicated and lengthy, and is omitted here. Interested readers may refer to these papers. The steps involved in these methods are listed below.

Assume ψ_0 is a plane wave whose direction is represented by a unit vector θ, that is, $\psi_0 = \exp(-\mathrm{i}k_0\theta \cdot \boldsymbol{r})$. Then, Gel'fand–Levitan's spectral inversion may be accomplished by

(1) Calculate the inner product for given spectral function ρ and $\rho_0(\rho | O = 0)$,

$$K(\boldsymbol{x}, \boldsymbol{y}) = (\psi_0(k_0, \boldsymbol{x}), d(\rho(k_0) - \rho_0(k_0))\psi_0(k_0, \boldsymbol{y}))\,. \tag{149}$$

(2) Solve the 3D counterpart of Gel'fand–Levitan integral equation for $h(\boldsymbol{x}, \boldsymbol{y})$

$$h(\boldsymbol{x}, \boldsymbol{y}) = K(\boldsymbol{x}, \boldsymbol{y}) - \int_{|z|<|x|} h(\boldsymbol{x}, \boldsymbol{z})K(\boldsymbol{z}, \boldsymbol{y})\,\mathrm{d}\boldsymbol{z} \quad \text{for } |x| > |y|\,. \tag{150}$$

(3) Calculate the Radon transform:

$$w(\alpha, \theta, x) = \int h(x, y)\delta(\alpha - \theta \cdot y)\, \mathrm{d}y \,. \tag{151}$$

(4) Finally, the potential function is obtained by

$$O(x) = -2\theta \cdot \nabla[w(\theta \cdot x^{+}, \theta, x) - w(\theta \cdot x^{-}, \theta, x)] \,. \tag{152}$$

For inverse scattering problems, the spectral function may be constructed from the S matrix via the Jost function. A more direct method is Marchenko's method which may be implemented by the following procedure:

(1) Calculate the Fourier transform:

$$K(\alpha, \theta, \theta') = [\mathrm{i}/(2\pi)^2] \int_{-\infty}^{\infty} kA(k, -\theta, \theta') \exp(-\mathrm{i}k\alpha)\, \mathrm{d}k \,, \tag{153}$$

where the far field scattering amplitude $A(k, \theta', \theta)$ is a normalized observation at direction θ' for the above setting.

(2) Solve the generalized Marchenko equation for w

$$w(\alpha, \theta, x) = \int K(\alpha + x \cdot \theta', \theta, \theta')\, \mathrm{d}\theta' + \int \mathrm{d}\theta' \int_{x \cdot \theta'}^{\infty} K(\alpha + \beta, \theta, \theta') w(\beta, \theta', x)\, \mathrm{d}\beta \,. \tag{154}$$

(3) Calculate the potential function

$$O(x) = 2\theta \cdot \nabla w(\theta \cdot x^{+}, \theta, x) \,. \tag{155}$$

Unfortunately, the authors are unaware of any applications or any numerical results using these methods.

4. Conclusions

From the above discussion, it is clear that one must solve an integral equation if an exact solution is preferred. With today's computer technology, this would be computationally expensive, and possibly prohibitive for multi-dimensional inverse scattering. Nevertheless, exact methods may have considerable potential in the future.

For the present time, an approximation seems necessary in order to be practical. In the framework of crack models in NDT, a generalization of DWBA seems appropriate and possible. The implementation of such a method would be quite feasible. To approach more general problems, the far field scattering amplitude may be employed by means of an approximation technique, such as a hologram.

References

[1] R.G. Newton, Inversion of reflection data for layered media: a review of exact methods, Geophys. J. R. Astr. Soc. 56 (1981) 191.

[2] J.P. Jones, Impediography: A new approach to noninvasive ultrasonic diagnosis, the 25th Annual Conference on Engineering in Medicine and Biology, Oct. 1972.

[3] J.P. Jones, Impediography: a new ultrasonic technique for nondestructive testing and medical diagnosis, Proc. First Int. Conf. on Ultrasonics, London, England, March 1973.
[4] S.M. Candel, F. Defillipi et al., Determination of the inhomogeneous structure of a medium from its plane wave reflection response, Part II: A numerical approximation, J. Sound and Vibration 68 (1980).
[5] H. Wright, Impulse response function corresponding to reflection from a region of continuous impedance change, J. Acoust. Soc. of Am. 53 (1973).
[6] J.R. Wait, Electromagnetic Waves in Stratified Media (Pergamon Press, Oxford, 1962).
[7] J.P. Jones, Further experiments with impediography, a new ultrasonic technique with biomedical implications, the 84th Meeting of Acoust. Soc. of Am., Nov. 1972.
[8] J.P. Jones, In vitro visualization of liver metastases using ultrasonic impediography, the 86th Meeting of Acoust. Soc. of Am., Nov. 1973.
[9] J.R. Taylor, Scattering Theory (Wiley & Sons, New York, 1972).
[10] W. Tobocman, Iterative inverse scattering method employing Gram–Schmidt orthogonalization, J. Comput. Physics, No. 64 (1986) 230–245.
[11] K. Santosh, W. Tobocman et al., in vivo biomicroscopy with ultrasound, Ultrasonics 25 (1987) 274–282.
[12] P.L. Goupillaud, An approach to inverse filtering of near-surface layer effects from seismic records, Geophysics 24(6) (1961) 754–760.
[13] J.A. Ware and K. Aki, Continuous and discrete inverse-scattering problems in a stratified elastic medium. I. plane waves at normal incidence, J. Acoust. Soc. Amer., 45(4) (1968) 911–921.
[14] P.C. Pedersen, O.J. Trekiak et al., Numerical techniques for the inverse acoustical scattering problem in layered media, Acoustical Imaging 12 (1982) 443.
[15] I. Gel'fand and B. Levitan, On the determination of a differential equation from its spectral function, Izv. Akad. Nayk. SSSR Ser. Mat. 2, No. 15 (1951) 309–360; Amer. Math. Soc. Transl. 2, No. 1 (1955) 253–304.
[16] V. Ambarzumian, Uber eine frage der eigenwerttheorie, Z. Physik 53 (1929) 690–695.
[17] V. Marchenko, Certain questions of the theory of a second-order differential operator, Dokl. Akad. Nauk SSSR (N.S.) 72 (1950) 457–460.
[18] M. Krein, Solution of the inverse Sturm–Liouville problem, Dokl. Akad. Nauk SSSR (N.S.) 76 (1951) 21–24.
[19] Z.S. Agranovich and V.A. Marchenko, The Inverse Problem of Scattering Theory (Gordon and Breach, New York, 1963).
[20] A. Alskseev, Some inverse problems of the theory of wave propagation, I and II, Izv. Akad. Nauk SSSR, Ser. Geofiz. 11 (1962) 1515–1531.
[21] A. Alskseev, Inverse dynamic problems of seismology, in some methods and algorithms for the interpretation of geophysical data, Moscow: Izd. Nauka (1967) 9–48.
[22] R. Carrol and F. Santosa, Scattering techniques for a one dimensional inverse problem in geophysics, Math. Meth. Appl. Sci. 3 (1981) 145–171.
[23] F. Santosa, Numerical scheme for the inversion of acoustical impedance profile based on the Gelfand–Levitan method, Geophys. J. R. Astr. Soc. 70 (1982) 229–243.
[24] W. Symes, Inverse boundary value problems and a theorem of Gel'fand and Levitan, J. Math. Anal. Appl. 71 (1979) 379–402.
[25] R. Burridge, The Gelfand-Levitan, the Marchenko, and the Gopinath-Sondhi integral equations of inverse scattering theory, regarded in the context of inverse impulse-response problems, Wave Motion 2 (1980) 305–323.
[26] G. Balanis, Inverse scattering determination of inhomogeneities in sound speed, J. Math. Phys. 23 (1982) 2562–2568.
[27] G. Balanis, The Plasma Inverse Problem, J. Math. Phys. 13 (1972) 1001–1005.
[28] H. Kritikos, D. Jaggard et al., Numerical reconstruction of smooth dielectric profiles, Proc. IEEE 70 (1982) 245–247.
[29] D. Jaggard and K. Olson, Numerical reconstruction for dispersionless refractive profiles, J. Opt. Soc. Am. A2 (1985) 1931–1985.
[30] D. Jaggard and P. Frangos, The electromagnetic inverse scattering problem for one-dimensional dispersionless dielectrics, IEEE Trans. Antennas Propagat. AP-35 (1987) 934–946.
[31] D. Ge, D. Jaggard et el., Perturbational and high frequency profile inversion, IEEE Trans. Antennas Propagat. AP-31 (1983) 803–804.
[32] D. Jaggard and Y. Kim, Accurate one-dimensonal inverse scattering using a nonlinear renormalization technique, J. Opt. Soc. Am. A2 (1985) 1922–1930.
[33] P. Frangos and D. Jaggard, The reconstruction of stratified dielectric profiles using successive approximations, IEEE Trans. Antennas Propagat. AP-35 (1987) 1267–1272.
[34] R. Courant, E. Isaacson and M. Rees, On the solution of nonlinear Hyperbolic Differential Equations by Finite Differences, Comm. Pure App. Math. Vol. V (1952) 243–255.
[35] F. Santosa, H. Schwetlick, The inversion of acoustical impedance profile by methods of characteristics, Wave Motion 4 (1982) 99–110.

[36] H. Schwetlick, Inverse methods in the reconstruction of acoustical impedance profile, J. Acoust. Soc. Am 73 (1983) 1179–1186.
[37] H. Schwetlick, W. Sachse, Stable impedance profile reconstruction by spike train approximation, Acoustical Imaging 13 (1983) 247.
[38] G.T. Herman and S.W. Rowland, Three methods for reconstructing objects from X-rays: a comparative study, Computer Graphics and Image Processing 2 (1973) 151.
[39] T.S. Huang, M. Kaveh and S. Berger, Some further results in iterative image restoration, presented at the Annual Meeting of the Optical Society of America, Boston, MA, Oct. 1975.
[40] C. Lanczos, Linear Differential Operators (London, England: Van Nostrand, 1964).
[41] R. Gordon, R. Bender and G.T. Herman, Algebraic Reconstruction Technique (ART) for three-dimensional electron microscopy and X-ray photography, J. Theor. Biol. 29 (1970) 471.
[42] S.H. Bellman, R. Bender, R. Gordon and J.E. Rowe, ART in science, being a defense of algebraic reconstruction techniques for three-dimensional electron microscopy, J. Theor. Biol. 32 (1971) 205.
[43] G.T. Herman and S. Rowland, Resolution in ART: an experimental investigation of the resolving power of an algebraic picture reconstruction technique, J. Theor. Biol. 33 (1971) 213.
[44] G.N. Ramachandran and A.V. Lakshminarayanan, Three-dimensional reconstruction from radiographs and electron micrographs: II. Application of convolutions instead of Fourier transforms, Proc. Nat. Acad. Sci. 68, No. 9, p. 2236.
[45] G.N. Ramachandran and A.V. Lakshminarayanan, Three-dimensional reconstruction from radiographs and electron micrographs: III. Description and application of the convolution method, Indian J. Pure Appl. Phys. 9 (1971) 997.
[46] D.J De Rosier and A. Klug, Reconstruction of three-dimensional structures from electron micrographs, Nature 217 (Jan. 13, 1968) 130.
[47] D.W. Sweeney and G.M. Vest, Reconstruction of three-dimensional refractive index fields from multidimensional interferometric data, Appl. Optics 2 (1973).
[48] M. Slaney, A.C. Kak and L. Larsen, Limitations of imaging with first-order diffraction tomography, IEEE Trans. Microwave Theory Tech., vol. MTT-32, no. 8 (1984) 860.
[49] M. Soumekh, M. Kaveh, A theoretical study of model approximation error in diffraction tomography, IEEE Trans. Ultrason. Ferro. Freq. Contr., vol. IFFC-33, no. 1 (1985) 10.
[50] M. Kaveh, M. Soumekh and R.K. Mueller, A comparison of Born and Rytov approximations in acoustic tomography, Acoustical Imaging 11 (1981).
[51] A.J. Devaney, Inverse scattering theory within the Rytov approximation, Opt. Lett. 6 (1981) 374.
[52] S.X. Pan and A.C. Kak, A computational study of reconstruction algorithms for diffraction tomography: interpolation versus filtered backpropagation, IEEE Trans. Acoust., Speech and Signal Processing, vol. ASSP-31, no. 5 (1983) 1262.
[53] R.K. Mueller, M. Kaveh and G. Wade, Reconstructive tomography and applications to ultrasonics, Proc. IEEE, vol. 67 (1979) 567.
[54] S.K. Kenue and J.F. Greenleaf, Limited angle multifrequency diffraction tomography, IEEE Trans. Sonics Ultrasonics, vol. SU-29 (1982) 213.
[55] D. Nahamoo, S.X. Pan and A.C. Kak, Synthetic aperture diffraction tomography and its interpolation-free computer implementation, IEEE Trans. Sonics Ultrasonics, vol. SU-31, no. 4 (1984) 218.
[56] A.J. Devaney, A filtered backpropagation algorithm for diffraction images, Ultrasonic Imaging 4 (1982) 336.
[57] B.A. Roberts and A.C. Kak, Reflection mode diffraction tomography, Ultrasonic Imaging 7 (1985) 300.
[58] S.A. Johnson and M.L. Tracy, Inverse scattering solutions by a sinc basis, mutiple source, moment method – Part I: Theory, Ultrasonic Imaging 5 (1983) 361.
[59] F. Stenger, Numerical methods based on Whittaker cardinal, or sinc functions, SIAM Review 23, No. 2 (1981) 165.
[60] J. Richmond, Scattering by a dielectric cylinder of arbitrary cross-sectional shape, IEEE Trans. Antennas Propagat. AP-13 (1965) 334.
[61] M.L. Tracy and S.A. Johnson, Inverse scattering solutions by a sinc basis, multiple source, moment method – Part II: Numerical evaluations, Ultrasonic Imaging 5 (1983) 376.
[62] S.A. Johnson, Y. Zhou and M.L. Tracy, Inverse scattering solutions by a sinc basis, multiple source, moment method – Part III: Fast algorithms, Ultrasonic Imaging 5 (1983) 376.
[63] T.J. Cavicchi, Diffraction tomography and the sinc basis moment method, Ph.D. Thesis of University of Illinois at Urbana-Champaign (1988).
[64] R.G. Newton, Inverse scattering. II. Three dimensions, J. Math. Phys. 21 (1980) 1698; 22 (1981) 631; 23 (1982) 693.
[65] R.G. Newton, Inverse scattering. III. Three dimensions, continued, J. Math. Phys. 22 (1981) 2191; 23 (1982) 693.
[66] R.G. Newton, Inverse scattering. IV. Three dimensions: Generalized Marchenko construction with bound states, J. Math. Phys. 23 (1982) 594.