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A Comparison of Absorbing Boundary Conditions (ABCs) to Perfectly Matched Layers (PML) in the 2D Frequency Domain Finite Element Method

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Abstract: In this paper we examine the effectiveness of ABCs and PML in truncating FEM mesh in polar coordinates. The model problem is based on the scattering of a plane wave by a conducting cylinder. The error is measured using L_2 norm on a surface.

1. Introduction

Unbounded problems have been a challenge to computation with Finite Element Method (FEM) to solve problems such as radiation/scattering. One must prevent outgoing waves from reflecting from an artificial boundary introduced in order to limit the number of unknowns. Because of the importance of the subject many approaches have been proposed and the research is ongoing.

The most common approach is to define boundary operators on the artificial boundary. In some sense, the operators are designed to replace the Sommerfeld radiation condition. Over the past years approximated Absorbing Boundary Conditions (ABCs) have received most of the attention from the FEM community . This is justified by the fact that these ABCs are local operators that preserve the sparsity of the stiffness matrix. In computational electromagnetics two classes of ABCs have proven especially important. They are those proposed by Bayliss-Turkel and Engquist-Majda [1].

The Perfectly Matched Layer (PML) is a new approach that has been the focus of extensive research in this area of mesh truncation. This new technique can be interpreted as a lossy medium, the PML, that can be matched to the interior domain for all frequencies and all angles less than the grazing incidence. Since the mesh must extend to the PML region it also has to be truncated. However, the problem becomes simpler once the wave amplitude is supposed to decay very fast.

Our goal in this paper is to investigate the effectiveness of ABCs versus PML in truncating the domain in FEM solutions of the scalar Helmholtz equation. To this end, we first present a brief summary of a model problem and the schema used to truncate the domain. Then the model problem is examined for a plane wave scattered by a cylinder. Finally results and conclusions are presented.

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2. Two Dimensional Model Problem

Assume a scattering problem with z-invariant electromagnetic field. In this case, the governing equation is the scalar Helmholtz equation

$$\nabla \cdot \nabla u + k^2 u = 0 \tag{1}$$

where u denote either the z component of the electric or magnetic field and $k=\omega\sqrt{\mu_o \varepsilon_o}$. We define $u = u_i + u_s$ where u_i is a given incident wave and u_s is the unknown scattered field.

For scatterers bounded by perfect conductors the boundary conditions are $u = 0$ for E_z and $\partial u/\partial n = 0$ for H_z on the boundary of the scatterer, Γ_s. The behavior of u_s at infinity is specified by the Sommerfeld radiation condition.

To solve this problem using FEM the unbounded domain is truncated by an artificial boundary, Γ_a. Next we assume that the Sommerfeld condition can be replaced by

$$\frac{\partial u_s}{\partial \rho} + \mathrm{B} u_s = 0 \qquad \text{on } \Gamma_a \tag{2}$$

where B is an operator. In this situation the following weak form is used to find an approximation for u in the finite element framework [2]

$$\int_{\Omega_c} (\nabla v \cdot \nabla u_s - k^2 v u_s) d\Omega + \int_{\Gamma_a} v \mathrm{B} u_s d\Gamma = -\int_{\Gamma_s} v \frac{\partial u_i}{\partial n} d\Gamma \tag{3}$$

where v is the test function. Our goal is to examine the error introduced in the FEM solution due to approximations enforced by Γ_a.

3. Mesh Truncation

We now consider the techniques ABCs and PML to truncate the FEM mesh.

3.1 Summary of 2D ABCs

We selected the two most widely used ABCs for computational electromagnetics. They can be written as

$$\frac{\partial u}{\partial \rho} = \alpha(\rho) u + \beta(\rho) \frac{\partial^2 u}{\partial \theta^2} \qquad \text{on } \Gamma_a \tag{4}$$

The most common ABC on circular boundaries is the second order Bayliss and Turkel condition (BT2). Written in the form (4) it corresponds to

$$\alpha(\rho) = \frac{k^2 \rho - 3/8\rho - j3k/2}{1 + jk\rho} \quad \text{and} \quad \beta(\rho) = \frac{1}{2\rho(1 + \mathrm{jk}\rho)} \tag{5}$$

Another well known sequence of ABCs are those proposed by Engquist-Majda [3]. Their second-order condition (EM2) at a circular boundary can be written as (4) where

$$\alpha(\rho) = \frac{-1 - j2k\rho}{2\rho} \quad \text{and} \quad \beta(\rho) = \frac{1 - \mathrm{jk}\rho}{2\mathrm{k}^2 \rho^3} \tag{6}$$

3.2 PML in Polar Coordinates

We consider now the perfectly matched layer used as a means to truncate the FEM mesh. This new technique can be interpreted as a special lossy material boundary layer that is reflectionless for all frequencies and angles. The approach used to model the PML is the complex stretched coordinates as presented in [4]. Using the complex radius

$$\tilde{\rho} = \rho - \frac{j}{\omega}\int_{\rho_o}^{\rho} \sigma(\rho')d\rho' = \rho s_\rho^* \quad \text{and} \quad s_\rho = \frac{\partial \tilde{\rho}}{\partial \rho} = 1 - j\frac{\sigma(\rho)}{\omega} \tag{7}$$

equation (1) in polar coordinates in the PML region can be expressed as

$$\nabla \cdot \Lambda \nabla u + k^2 s_\rho s_\rho^* = 0 \tag{8}$$

where

$$\Lambda = diag\left\{ \frac{s_\rho^*}{s_\rho} \quad \frac{s_\rho}{s_\rho^*} \right\} \tag{9}$$

To apply the FEM we rewrite this equation back in rectangular coordinates using rotational transformations [4].

4. Numerical Results

The computational domain for the problem of scattering by a circular boundary is shown in Fig. 1. The scatterer has radius a . To design the mesh the relation λ/h>20 is applied in the radial and axial direction, and the elements are linear. The PML region is defined in the annulus $\rho_0<\rho<\rho_a$ with parameters s=1-j2 and s=1-j2(ρ-ρ₀)/ρ . The homogeneous Dirichlet condition is imposed on the external boundary of the PML. To solve the matrix equations the Bi-Conjugate Gradient (BiCG) method is used.

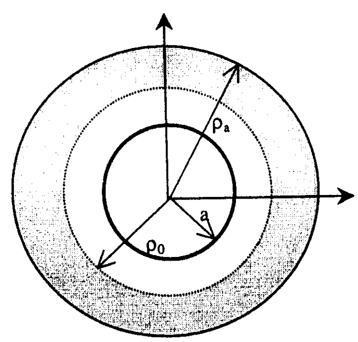

Fig1: Geometry of the computational domain.

The far-field can be calculated using the series expansion

$$u^s(\rho,\theta) = \sum_{n=-\infty}^{\infty} a_n H_n^{(2)}(k\rho)e^{jn\theta} \tag{10}$$

where $$a_n = \frac{1}{2\pi H_n^2(kR)} \int_{-\pi}^{\pi} u(R,\theta) e^{-jn\theta} R d\theta \qquad (11)$$

H_n^2 is the cylindrical Hankel function of the second kind and R is the radius where we sample the FEM solution. The series is truncated with n=10.

The error can be measured using L_2-norm on the surface that is

$$\|\hat{u} - u\|^2 = \int_0^{2\pi} |\hat{u}(R,\theta) - u(R,\theta)|^2 \, R d\theta \qquad (12)$$

$\hat{u}$ corresponds to the FEM solution.

In Tables 1 and 2 we list in column A the errors calculated using (12). In the same tables the number of iterations for convergence of the BiCG are shown in column B.

Tab. 1: Comparison when Γ_a is at ρ=a+0.5λ and ρ_0=a+0.25λ

ka	EM2		BT2		PML	
	A	B	A	B	A	B
1	0.0008	106	0.0013	100	0.0025	132
5	0.0050	182	0.0042	181	0.0011	235
10	0.5720	368	0.5603	366	0.2354	289

Tab. 2: Comparison when Γ_a is at ρ=a+λ and ρ_o=a+0.5λ

ka	EM2		BT2		PML	
	A	B	A	B	A	B
1	4.90E-4	254	5.96E-4	251	0.0022	307
5	6.44E-4	356	7.23E-4	3 59	7.87E-4	484
10	0.2824	933	0.2814	633	0.2643	717

The results show that comparative to ABCs, PML does not perform well for low frequency ka=1. On the other hand, for high frequency the solution with PML becomes more accurate for boundary closer to the source. Also the convergence of BiCG for PML improves in relation to ABCs for ka=10. The position of the artificial boundary affects the accuracy of ABC solutions more than PML solutions, mainly for ka=1 and ka=5.

5. Conclusions

In this paper we have presented a comparison of the PML to second order Bayliss-Turkel and Engquist-Majda conditions in polar coordinates. In order to use ABCs, we might put the artificial boundary far from the source to improve accuracy. For the problem shown PML performs better than ABCs when the frequency increases.

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