Reduction of a Finite-Element Parametric Model Using Adaptive POD Methods—Application to Uncertainty Quantification

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Model order reduction methods enable reduction of the computation time when dealing with parametrized numerical models. Among these methods, the proper orthogonal decomposition method seems to be a good candidate because of its simplicity and its accuracy. In the literature, the offline/online approach is generally applied but is not always required especially if the study focuses on the device without any coupling with others. In this paper, we propose a method to adaptively construct the reduced model while it limits the evaluation of the full model when appropriate. A stochastic magnetostatic example with 14 uncertain parameters is studied by applying the Monte Carlo simulation method to illustrate the proposed procedure. In that case, it appears that the complexity of this method does not depend on the number of input parameters and so is not affected by the curse of dimensionality.

Index Terms—**Error estimation, finite-element method (FEM), model order reduction (MOR), proper orthogonal decomposition (POD), uncertainty quantification.**

I. Introduction

IN SIMULATION-BASED design or in uncertainty quantification, parameterized models are needed. In order to obtain good accuracy, numerical models based on the finite-element method (FEM) are often used. The issue is then the computation time that can be very long, especially if the number of parameters is high, more than about a dozen. Recently, model order reduction (MOR) methods, like the proper orthogonal decomposition (POD) method or the reduced basis (RB) method, have been applied to computational electromagnetics for uncertainty quantification or design [1], [2]. An approximation of the full parametrized model (the original FE model) is then sought in a space spanned by an RB, which enables one to reduce the number of degrees of freedom and consequently the computation time by reducing the size of the equation system [3]–[5]. However, the accuracy of the reduced model is strongly related to the choice of the RB, which is obtained from the solutions, so-called snapshots, of the full problem for particular input parameter sets. The offline/online approach is generally used to construct an accurate reduced model [6], [7]. During the offline stage, the whole parameter space is spanned in order to select the best snapshots using a greedy algorithm based on an error estimator, which evaluates the error of reduction [8], [12]. During the online stage, the reduced model is evaluated intensively. This approach is interesting to develop a generic model incorporated as a building block within a bigger system. However, if the study holds only on the device by itself without any coupling with other models like, for example, during a design process or a robustness analysis based on a Monte Carlo simulation, this decoupling between the two stages is not always justified.

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In this paper, we propose a procedure to construct adaptively the reduced model, while its utilization that limits the evaluations of the full model when appropriate. First, the POD technique is briefly presented. Then, the proposed procedure is described. We introduce a simple error estimator based on [9], which bounds the error due to reduction. Finally, the proposed procedure is applied in the case of a stochastic magnetostatic problem, whose geometry is defined by 14 parameters.

II. Reduction of a Parametric Model Using the POD Method

The discretization of a parameterized linear magnetostatic problem using the FEM on a domain $D(\mathbf{p})$ leads to the following linear system of equations:

$$\mathbf{S}(\mathbf{p})\mathbf{X}(\mathbf{p}) = \mathbf{F}(\mathbf{p}) \tag{1}$$

where $\mathbf{p} = (p_1, .., p_M)$ is the set of M input parameters, $\mathbf{S}(\mathbf{p})$ is the $N \times N$ stiffness matrix, $\mathbf{F}(\mathbf{p})$ is the $N \times 1$ source vector, $\mathbf{X}(\mathbf{p})$ is the $N \times 1$ vector of unknowns, and N is the number of degrees of freedom. The parameters can be related to the source terms such as the current, the material characteristics, or the physical dimensions as long as the connectivity between nodes and elements does not change. To account for the variation of the geometry of $D(\mathbf{p})$, the nodes are moved. To reduce the numerical errors and to avoid any overlapping of elements, an appropriate mapping $T(\mathbf{x}, \mathbf{p})$ ($\mathbf{x}$ is the position) is introduced, which transforms a mesh constructed on a reference domain D_{ref} (defined as the nominal parameter set values) into a mesh of the desired parameterized domain $D(\mathbf{p})$. The solution $\mathbf{X}(\mathbf{p})$ enables one to determine the field distribution and also the quantities of interest, which are usually either linear functions of $\mathbf{X}(\mathbf{p})$ (flux) or quadratic functions of $\mathbf{X}(\mathbf{p})$ (energy or force). If we denote G as the quantity of interest, which is a quadratic function of $\mathbf{X}(\mathbf{p})$, it can be written under the form

$$G = \mathbf{X}(\mathbf{p})^t \mathbf{G}(\mathbf{p})\mathbf{X}(\mathbf{p}) \tag{2}$$

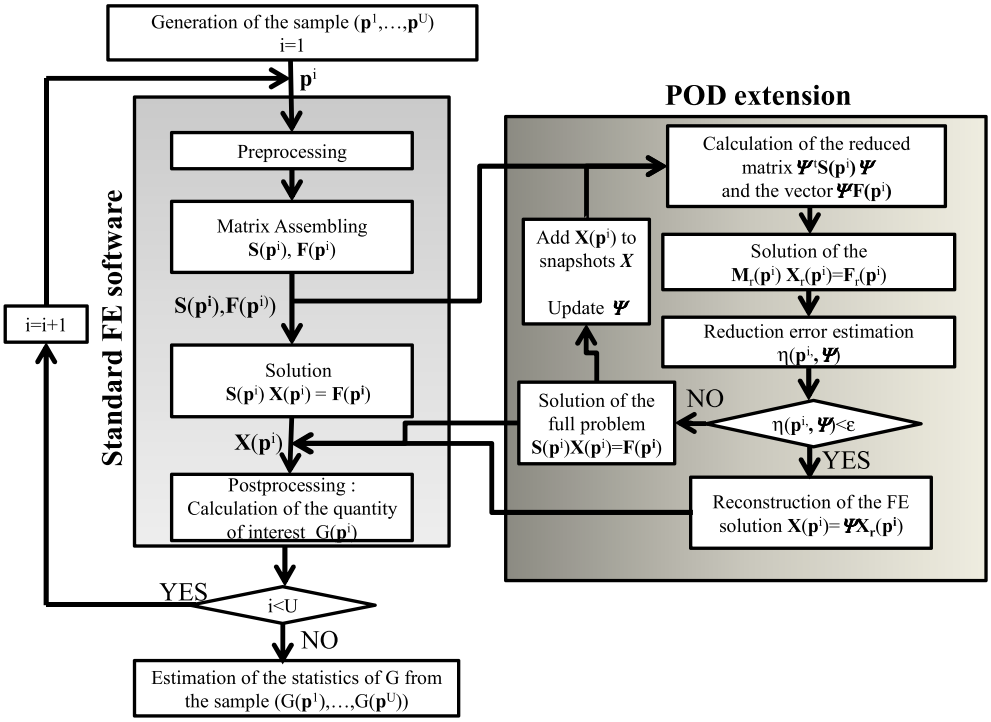

Fig. 1. Flowchart of the implantation of the adaptive POD extension along an FE standard software for uncertainty quantification using the MCSM (U is the size of the sample).

where $\mathbf{G}(\mathbf{p})$ is a $N \times N$ matrix. If models (1) and (2) are used in a design process or for uncertainty quantification, the number of solutions of the FE model can be very high for a large parameter number M due to the curse of dimensionality. To decrease the computation time, one should either limit the number of calls leading to a loss of accuracy or reduce the size of the linear system of (1) using an MOR method. The POD method, detailed in the following, is one of the most popular MOR methods. Consider the Z-parameter set realizations $(\mathbf{p}^1, \ldots, \mathbf{p}^Z)$ and the $N \times Z$ matrix X of the associated solutions $(\mathbf{X}(\mathbf{p}^1), \ldots, \mathbf{X}(\mathbf{p}^Z))$, so-called snapshots. We define the linear space K spanned by the vectors $(\mathbf{X}(\mathbf{p}^1), \ldots, \mathbf{X}(\mathbf{p}^Z))$. We denote the $N \times R$ matrix $\boldsymbol{\Psi}$ $(R \leq Z)$ of the vectors $(\boldsymbol{\Psi}^{\mathbf{1}}, .., \boldsymbol{\Psi}^{\mathbf{R}})$, an orthogonal basis of the space K. The matrix $\boldsymbol{\Psi}$ can be obtained by a singular value decomposition (SVD) of the matrix X. The idea of the POD method is to seek an approximation of the solution of (1) in the space K, which means that $\mathbf{X}(\mathbf{p})$ is approximated by the linear combination

$$\mathbf{X}(\mathbf{p}) \approx \boldsymbol{\Psi}\mathbf{X}_{\mathbf{r}}(\mathbf{p}) = \sum_{i=1}^{R} x_{ri}(\mathbf{p})\boldsymbol{\Psi}^i. \tag{3}$$

Replacing $\mathbf{X}(\mathbf{p})$ by the approximation (3) in (1), the equation system to be solved becomes overdetermined. By applying the Galerkin method, the vector $\mathbf{X}_{\mathbf{r}}(\mathbf{p})$ has to satisfy an $R \times R$ linear system of equations

$$\boldsymbol{\Psi}^t\mathbf{S}(\mathbf{p})\boldsymbol{\Psi}\mathbf{X}_{\mathbf{r}}(\mathbf{p}) = \boldsymbol{\Psi}^t\mathbf{F}(\mathbf{p}). \tag{4}$$

The size of the system (4) is then equal to R, which is much lower than N, the size of the full system (1). The solution of the system is much faster, i.e., reducing significantly the computation time even if the matrix $\boldsymbol{\Psi}^t\mathbf{S}(\mathbf{p})\boldsymbol{\Psi}$ is dense. From the solution $\mathbf{X}_r(\mathbf{p})$, an approximation of the FE solution can be derived $\mathbf{X}(\mathbf{p}) = \boldsymbol{\Psi}\mathbf{X}_r(\mathbf{p})$. The accuracy of the method is closely related to the choice of the parameters $(\mathbf{p_1}, \ldots, \mathbf{p_Z})$ used to determine the RB. It should be noted that the POD method can also be seen as a solver of an FE equation system. Indeed, it leads to an approximate solution $\boldsymbol{\Psi}\mathbf{X}_r(\mathbf{p})$ with the same format as the FE solution of the full problem (1). This means that the solution $\boldsymbol{\Psi}\mathbf{X}_r(\mathbf{p})$ is fully compatible with the FE postprocessor (Fig. 1).

III. Adaptive Procedure

In the literature, the POD method is often combined with an offline/online approach. During the offline stage, an iterative algorithm, called the greedy algorithm, is applied to determine the best snapshots. Suppose that, at the ith iteration of the offline stage, the snapshots have been determined for the parameter sets $(\mathbf{p}^1, \ldots, \mathbf{p}^i)$ and $\boldsymbol{\Psi}^{ii} = (\boldsymbol{\Psi}^{\mathbf{1}}, .., \boldsymbol{\Psi}^{\mathbf{i}})$, the corresponding RB. To determine the next snapshot $\mathbf{X}(\mathbf{p}^{i+1})$, we seek the parameter $\mathbf{p}^{i+1}$, which maximizes an error estimator $\eta(\mathbf{p}, \boldsymbol{\Psi}^{ii})$. The term $\eta(\mathbf{p}, \boldsymbol{\Psi}^{ii})$ estimates the error of the reduction process, i.e., the error between the solutions of the full model $\mathbf{X}(\mathbf{p})$ and of the reduced model $\mathbf{X}_r(\mathbf{p})$ without requiring the solution of the full problem. Such estimators are available in [6]–[8]. These are generally derived from the residual $\mathbf{R}(\mathbf{p})$ as

$$\mathbf{R}(\mathbf{p}) = \mathbf{F}(\mathbf{p}) - \mathbf{S}(\mathbf{p})\boldsymbol{\Psi}\mathbf{X}_{\mathbf{r}}(\mathbf{p}). \tag{5}$$

The crude one consists of solving the reduced problem for a sufficiently dense set of points S in the parameter

space and to select $\mathbf{p}^{i+1}$ equal to the point of S giving the maximum value of the estimator. Once $\mathbf{p}^{i+1}$ is determined, the full problem is solved and the solution $\mathbf{X}(\mathbf{p}^{i+1})$ is added to the snapshot set, and a new orthogonal basis $\boldsymbol{\Psi}^{ii+1}$ is derived by applying an SVD. The process is repeated until the error reaches a prescribed threshold. Then, the online stage concerns the many-queries parameters analysis stage where the reduced model is used intensively. This approach can be very useful when the reduced parametric model is used during the online stage as a building block of a model of a large system, because it can represent a device on a large parameter range with good accuracy while being very fast. However, if one wants just to evaluate once a quantity of interest for a sequence of parameter sets $\{\mathbf{p}^1..,\mathbf{p}^U\}$ like for uncertainty quantification or optimization, this dissociation in two stages is not necessarily required. Moreover, during the offline stage, the greedy algorithm requires the evaluation of the reduced models on the dense set S, whose size is affected by the curse of dimensionality. The idea is then to evaluate the quantity of interest G with the reduced model for each $\mathbf{p}^i$ and to evaluate the error of reduction. If the error of reduction is too significant, the full problem is then solved for the parameter set $\mathbf{p}^i$ and the snapshot set is updated, as well as the RB $\boldsymbol{\Psi}^i$. This method enables to evaluate the full model only when it is necessary. The algorithm of the proposed method has been given in Fig. 1 in the case of the Monte Carlo simulation method (MCSM) when the parameter sets, for which the model is evaluated, are defined *a priori*. We can notice that the number M of input parameters does not influence the algorithm complexity and so is not affected by the curse of dimensionality, preserving this property of the MCSM. One can note that the adaptive POD can be easily extended to any procedure in which the parameter sets are iteratively determined like in the case of a design procedure.

IV. Error Estimation

The previous procedure requires the evaluation of the error of reduction. Several error estimators derived from the residual have been proposed in the literature. In the following, an estimator is derived from [10], which is not necessarily efficient for large FE problems but provides error bounds. We suppose that the stiffness matrix $\mathbf{S}(\mathbf{p})$ is symmetric positive definite, which is the case with standard FE gaged potential formulations of static field problems. The error $e_{\text{red}}(\mathbf{p}, \boldsymbol{\Psi}^i)$ due to the reduction method is defined as

$$e^2_{\text{red}}(\mathbf{p}, \boldsymbol{\Psi}^i) = \Delta\mathbf{X}^t(\mathbf{p})\mathbf{S}(\mathbf{p})\Delta\mathbf{X}(\mathbf{p}) \tag{6}$$

where $\Delta\mathbf{X}(\mathbf{p})$ is a vector equal to $\mathbf{X}(\mathbf{p}) - \boldsymbol{\Psi}^i\mathbf{X}_r(\mathbf{p})$. The error is equal to zero when $\Delta\mathbf{X}(\mathbf{p})$ is equal to zero; that is to say, when the FE solution is equal to the solution given by the reduced problem $\boldsymbol{\Psi}^i\mathbf{X}_r(\mathbf{p})$. This error cannot be calculated in practice because it requires the calculation of the solution $\mathbf{X}(\mathbf{p})$ of the FE problem. However, the error can be expressed as a function of the residual [see (5)] as

$$e^2_{\text{red}}(\mathbf{p}, \boldsymbol{\Psi}^i) = \mathbf{R}^t(\mathbf{p})\mathbf{S}^{-1}(\mathbf{p})\mathbf{R}(\mathbf{p}). \tag{7}$$

This expression can also not be used in practice, because it requires the solution of $\mathbf{S}^{-1}(\mathbf{p})\mathbf{R}(\mathbf{p})$, which is of the size of the full problem. However, we can introduce the following error estimator:

$$\eta^2(p, \boldsymbol{\Psi}^i) = \mathbf{R}^t(\mathbf{p})\mathbf{S}^{-1}(\mathbf{p_0})\mathbf{R}(\mathbf{p}). \tag{8}$$

The parameter $\mathbf{p_0}$ is a well-chosen value of $\mathbf{p}$. The inverse of $\mathbf{S}(\mathbf{p_0})$ has to be calculated only once and then stored before the adaptive process. If the parameter $\mathbf{p}$ is related only to the reluctivity $\nu(\mathbf{x}, \mathbf{p})$ with $\mathbf{x}$ the position on D, we denote $\nu_{\min}(\mathbf{x})$ and $\nu_{\max}(\mathbf{x})$ such that

$$\nu_{\min}(\mathbf{x}) \leq \nu(\mathbf{x}, \mathbf{p}) \leq \nu_{\max}(\mathbf{x}) \quad \forall\mathbf{p} \quad \forall x \in D. \tag{9}$$

In the case of the vector potential formulation, it can be shown that [10]

$$k_{\min}\eta^2(\mathbf{p}, \boldsymbol{\Psi}^i) \leq e^2_{\text{red}}(\mathbf{p}, \boldsymbol{\Psi}^i) \leq k_{\max}\eta^2(\mathbf{p}, \boldsymbol{\Psi}^i) \tag{10}$$

with the coefficients $k_{\min} = \min_{x\in D}[\nu(\mathbf{x}, \mathbf{p_0})/\nu_{\max}(\mathbf{x})]$ and $k_{\max} = \max_{x\in D}[\nu(\mathbf{x}, \mathbf{p_0})/\nu_{\min}(\mathbf{x})]$. We can see that an error bound can be expressed as a function of the estimator. In the scalar potential formulation, an error bound can also be extracted while considering the permeability instead of the reluctivity. It should also be noted that the previous error bound property can be retrieved when the parameters are related to the geometry. In fact, changing the geometry of the domain $D(\mathbf{p})$ is equivalent to changing the permeability on a reference domain D_{ref} [9]. The permeabilities on the reference domain D_{ref} are then the functions of the dimensions and can be bounded as in (9).

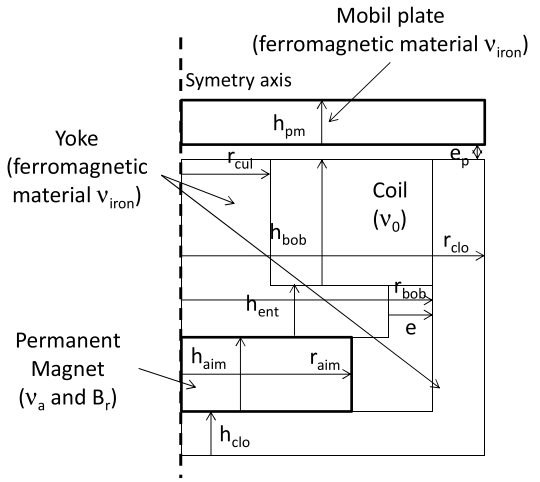

Fig. 2. Half of the geometry of the magnetic holder and the definition of the parameters (r_{cul}, r_{aim}, r_{bob}, r_{clo}, e, e_p, h_{clo}, h_{aim}, h_{ent}, h_{bob}, h_{pm}, $B_r\nu_{\text{iron}}$, ν_a).

V. Application

We consider a magnetic holder modeled by a 2-D FE vector potential formulation. The geometry of the device is defined as 14 parameters, which are represented in Fig. 2. The 11 parameters are dimensions and three are material characteristics. The quantity of interest is the force experienced by the mobile plate when the coil is not energized (only due to the permanent magnet). The force has been calculated using the Maxwell stress tensor. We have fixed nominal values for

TABLE I
ESTIMATION OF THE MEAN AND THE STANDARD DEVIATION FOR DIFFERENT VALUES OF THE ERROR CRITERION ε

Relative error ε	Full problem solutions	Mean (N)	Standard deviation (N)
$1{,}5{\cdot}10^{-2}$	1	77.87	7.549
$1{,}5{\cdot}10^{-4}$	6	78.11	8.743
$1{,}5{\cdot}10^{-6}$	16	78.11	8.857
0 (Full problem)	U	78.11	8.858

TABLE II
FIRST ORDER SOBOL INDICES (IN %) CALCULATED FROM THE FULL MODEL AND FROM THE ADAPTIVE PROCEDURE FOR DIFFERENT VALUES OF THE ERROR

Relative error ε	$1{,}5{\cdot}10^{-2}$	$1{,}5{\cdot}10^{-4}$	$1{,}5{\cdot}10^{-6}$	Full
r_{cul}	0.00	1.49	2.21	2.30
r_{aim}	0.00	3.15	3.77	3.60
r_{bob}	6.14	0.22	0.3	0.31
r_{clo}	30.5	15.9	14.3	14.6
e	0.29	1.49	1.58	1.60
e_p	0.46	5.92	6.92	7.08
h_{clo}	0.27	0.36	0.33	0.30
h_{aim}	9.38	17.8	16.4	16.2
h_{ent}	0.14	3.04	2.84	2.70
h_{bob}	0.06	0.92	0.85	0.91
h_{pm}	0.16	1.98	2.11	1.94
B_r	35.8	25.5	26.0	26.0
ν_{iron}	2.96	11.4	12.2	12.0
ν_a	13.3	9.31	9.14	9.10

the parameters p_i^{nom} and consider the parameter p^i as a uniform random variable in $[0.9\text{p}_i^{\text{nom}}, 1.1\text{p}_i^{\text{nom}}]$ except B_r, which has an interval of variation of $[0.95\text{B}_r^{\text{nom}}, 1.05\text{B}_r^{\text{nom}}]$. The FE mesh consists of 2846 nodes and 2750 rectangular elements. The modifications of geometry are considered by moving the nodes proportionally to the dimension variations (the transformation $T(\mathbf{x}, \mathbf{p})$ is a dilatation/contraction). We want to estimate the mean and the standard deviation of the force and also to determine the parameters that contribute the most to the variability of the force. Then, the most efficient way to lower the variability of the force is to reduce the variability of these parameters. This global sensitivity has been undertaken by estimating the Sobol indices. The MCSM has been used to estimate the sample mean and standard deviation and the Sobol indices, applying the methods proposed in [11]. We have applied the algorithm shown in Fig. 1 for a sample length U equal to 500. The entries of the parameter set $\mathbf{p_0}$ have been taken equal to the nominal values p_i^{nom}. Since the estimator is homogeneous to an energy, the relative value of ε has been defined versus the energy of the problem parameterized with $\mathbf{p_0}$. A criterion ε equal to zero means that the full problem is solved for each parameter set realization. The number of snapshots is then equal to U. In Table I, we have reported the number of solutions of the full problem (snapshots) as well as the estimation of the mean and the standard deviation for different values of the criterion ε for the estimator $\eta^2(\mathbf{p}, \boldsymbol{\Psi}^i)$ (see Section IV). As expected, we can see that the number of snapshots increases as ε decreases. Good accuracy on the mean and the standard deviation is obtained with six snapshots. In that case, the full problem has been solved six times and the remaining 494 points have been obtained using the reduced model with a maximum size equal to six. We have reported the Sobol indices calculated for different values of ε in Table II. We can see that the Sobol indices, even with low percentage, can be calculated accurately. The remanent magnetization B_r is the most influential parameter. A new MCSM has been launched with a smaller range for B_r of 2.5% instead of 5% while starting the procedure with the snapshots calculated during the first round and fixing an error of $\varepsilon = 1,510^{-6}$. The mean and the standard deviation of the force are equal to 78.07 and 8.09 N, respectively. This calculation does not require any additional call to the full model.

VI. CONCLUSION

In this paper, we have proposed an adaptive POD method to solve uncertainty quantification problems based on sampling techniques or optimization problems. The application shows that this method is promising. In future works, we propose to extend it in the case of an affine decomposition of the full parametric problem by using the RB Method.

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