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Error estimation in a stochastic finite element method in electrokinetics

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SUMMARY

Input data to a numerical model are not necessarily well known. Uncertainties may exist both in material properties and in the geometry of the device. They can be due, for instance, to ageing or imperfections in the manufacturing process. Input data can be modelled as random variables leading to a stochastic model. In electromagnetism, this leads to solution of a stochastic partial differential equation system. The solution can be approximated by a linear combination of basis functions rising from the tensorial product of the basis functions used to discretize the space (nodal shape function for example) and basis functions used to discretize the random dimension (a polynomial chaos expansion for example). Some methods (SSFEM, collocation) have been proposed in the literature to calculate such approximation. The issue is then how to compare the different approaches in an objective way. One solution is to use an appropriate *a posteriori* numerical error estimator. In this paper, we present an error estimator based on the constitutive relation error in electrokinetics, which allows the calculation of the distance between an average solution and the unknown exact solution. The method of calculation of the error is detailed in this paper from two solutions that satisfy the two equilibrium equations. In an example, we compare two different approximations (Legendre and Hermite polynomial chaos expansions) for the random dimension using the proposed error estimator. In addition, we show how to choose the appropriate order for the polynomial chaos expansion for the proposed error estimator. Copyright © 2009 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Applying the finite element method (FEM) to solve the Maxwell equations leads to valuable tools for understanding and predicting the features of electromagnetic devices. Currently, the input data to these tools (geometry of the device and material properties) are generally assumed to be known exactly. In some applications however, the input data are not known exactly and therefore, the

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random aspects of these data can be taken into account. In fact, mechanical parts are manufactured with dimensional tolerances, whereas some dimensions, such as air gaps in electric machines, are critical as they strongly influence the performance. Moreover, uncertainties in material composition, the characteristic changes with environmental factors (humidity, pressure, etc.) and thermal and mechanical inputs, which modify the electromagnetic behavior of the material with time, are often unknown. The material features might thus be considered as stochastic data. In addition, as the accuracy of the models used to describe the material behavior increases, the problem of repeatability for different samples of the same material becomes apparent. In practice, if material properties are non-repetitive, the normal process of increasing the precision of the deterministic model becomes futile. Consequently, stochastic models (having stochastic inputs and outputs) are better suited in taking into account the uncertainties in model dimensions and material properties.

During the last decade or so, significant effort has been invested in solving partial differential equations that govern many physical processes that take into account random behavior laws and external inputs (such as forces on boundaries). In the available literature, two classes of methods have been proposed to solve such a system of equations [1, 2]. The 'non-intrusive' methods are embedded deterministic numerical models in an environment of stochastic procedures. Among the available techniques, the Monte Carlo Simulation Method is probably the best known and widely used in different scientific areas (financial mathematics, biostatistics, mechanics, etc.) [3]. The Monte Carlo method is robust and simple but very time consuming especially when coupled with a finite element model. Other methods have been proposed in order to reduce the computation cost, for example, Stochastic Surface Response Method and Collocation Methods.

The second class of methods is the so-called 'intrusive' methods that require specific developments to solve the stochastic problem. The *mean-centered perturbation method* consists of expanding the unknown field around its mean. This approach is very useful in determining the first- and second-order statistical moments (mean and variance) of the unknown field. However, the extension to moments of higher orders is very difficult and time consuming. In addition, it is not very accurate for problems with input data with large deviations. At the beginning of the 1990s, Ghanem and Spanos proposed the so-called Spectral Stochastic FEM that consists of discretizing the unknowns in the spatial and stochastic domains simultaneously [4]. Other schemes of discretization have followed [5, 6].

Spectral stochastic finite element models have been applied in various domains of physics including mechanics [7], fluid dynamics [8] and electromagnetics [9–11]. A large variety of methods exists and for a given method one can employ many different approaches. For example, the stochastic polynomial decomposition used to approximate the 'random dimension' is often used either for non-intrusive or intrusive methods. Different types of polynomial expansions are available. Comparisons between these expansions have been done on academic examples but not on more realistic devices. These comparisons need to be done to obtain the most suitable expansions for the best compromise of accuracy versus computation time. In that case, the analytical solution is not available; hence, the error due to discretization must be estimated. *A priori* error estimations have been proposed to compare different approximation schemes [2]. In [12], an *a posteriori* error estimator based on the evaluation of the constitutive relation error has been proposed for stochastic problems in mechanics. It is in fact an extension of an error estimator based on the Prager–Synge theorem [13]. The main feature in this estimator is that it estimates a distance from the exact solution of the problem without knowing the exact solution.

In this paper, we develop an *a posteriori* error estimator based on the constitutive relation error in static electromagnetism. The calculation of the error estimation requires two fields that

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satisfy the equilibrium equations. These fields are obtained by solving two complementary potential formulations. After a short introduction on notations in Section 2, we present the deterministic problem and then discuss the error estimation procedure in Section 3. We also present the solution of two complementary potential formulations to obtain the admissible required solutions for the error estimation. Section 4 presents the stochastic problem and the proposed estimator and its properties in the stochastic case. Then, we detail the calculation of the error when the approximated fields are given under the form of an expansion of orthogonal polynomials. Finally, in Section 5, the estimated error is used to compare two polynomial approximations based on the Hermite and Legendre polynomials.

2. NOTATIONS

In the following we consider a contractible domain D in R^3 with a boundary S. Let the standard inner scalar product defined on D be: $(\mathbf{X},\mathbf{Y})_D=\int_D \mathbf{X}\cdot\mathbf{Y}\,\mathrm{d}D$ with X and Y functions in $D\to R$ or $D\to R^3$. Finally, let $L^2(D)$ and $\mathbf{L}^2(D)$ denote the Hilbert spaces of finite energy functions. We denote $L^2_{\mathrm{grad}}(D),\mathbf{L}^2_{\mathrm{curl}},\mathbf{L}^2_{\mathrm{div}}$ spaces such that;

$$\begin{aligned} L^2_{\mathrm{grad}} &= \{u\in L^2(D) \text{ and } \mathbf{grad}\ u\in\mathbf{L}^2(D)\}\\ \mathbf{L}^2_{\mathrm{curl}} &= \{\mathbf{u}\in\mathbf{L}^2(D) \text{ and } \mathbf{curl}\ \mathbf{u}\in\mathbf{L}^2(D)\}\\ \mathbf{L}^2_{\mathrm{div}} &= \{\mathbf{u}\in\mathbf{L}^2(D) \text{ and } \mathrm{div}\,\mathbf{u}\in L^2(D)\} \end{aligned} \tag{1}$$

We also denote Ω the set of outcomes θ, F the set of events, P the probability measure that is a real function from F in $[0,1]$. We consider a real random variable X. The expectation is written as:

$$E[X]=\int_\Omega X\,\mathrm{d}P \tag{2}$$

If the random variable X has a probability density function $\rho: R\to[0,+\infty]$ then:

$$E[X]=\int_R x\rho(x)\,\mathrm{d}x \tag{3}$$

$L^2(\Omega)$ is the space of random variables with a finite variance (i.e. $E[X^2]$ exists).

3. DETERMINISTIC ELECTROKINETIC PROBLEM

3.1. Description of the problem

Consider a contractible domain D (without loops and holes) with S the surface of the domain D. If $\mathbf{E}\in\mathbf{L}^2_{\mathrm{curl}}$ is the electric field and $\mathbf{J}\in\mathbf{L}^2_{\mathrm{div}}$ the current density, the electrokinetic problem can be written as follows:

$$\mathbf{curl}\ \mathbf{E}=\mathbf{0} \tag{4}$$

$$\mathrm{div}\,\mathbf{J}=0 \tag{5}$$

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The boundary conditions on the surface S are:

$$\mathbf{E}\wedge\mathbf{n}=\mathbf{0} \quad \text{on } S_E \tag{6}$$

$$\mathbf{J}\cdot\mathbf{n}=0 \quad \text{on } S_J \tag{7}$$

S_E and S_J are surfaces such that $S_E\cap S_J=0$ and $S_E\cup S_J=S$. Electromagnetic field sources are either prescribed fluxes of the current density $\mathbf{J}$ flowing through sub-surfaces of S_E or circulations of the electric field between two distinct surfaces of S_E. For the sake of simplicity, we assume that there is only one source term but the discussion can be easily extended to several source terms as we will see in an example in Section 6 [14]. The surface S_E is split into two distinct boundaries S_{E1} and S_{E2}. We assume that the circulation of the electric field between S_{E1} and S_{E2} is prescribed as V. The fields $\mathbf{E}$ and $\mathbf{J}$ are linked by the constitutive relation:

$$\mathbf{J}=\sigma(x)\,\mathbf{E} \tag{8}$$

With $\sigma(x)$ the conductivity which is a real, strictly positive function of the position x defined on D. In most deterministic problems in electrokinetics, the conductivity is a piecewise constant function on N subdomains D_i of D:

$$\sigma(x)=\sum_{i=1}^{N}\sigma_i I_i(x) \tag{9}$$

With σ_i a strictly positive real value and $I_i(x)$ an index function associated with the subdomain D_i ($I_i(x)=1$ on D_i and 0 elsewhere). In the following, the exact solution of the above problem is denoted as $(\mathbf{E}_{\text{ex}},\mathbf{J}_{\text{ex}})$.

3.2. Error estimation

Solving the previous problem using a numerical method such as the FEM yields an approximate solution. An error estimator can be used to assess the quality of the solution. Consider a pair of fields $(\mathbf{E}_{\text{ad}},\mathbf{J}_{\text{ad}})$ such that $\mathbf{E}_{\text{ad}}$ satisfies (4) and (6) and $\mathbf{J}_{\text{ad}}$ satisfies (5) and (7). As the conductivity is a strictly positive value on the whole domain D, we can define two scalar products $(,)_\sigma$ and $(,)_{\sigma-1}$ from the standard scalar product defined on D:

$$(\mathbf{X},\mathbf{Y})_\sigma=(\mathbf{X},\ \sigma\mathbf{Y})_D \tag{10}$$

$$(\mathbf{X},\mathbf{Y})_{\sigma-1}=(\sigma^{-1}\mathbf{X},\ \mathbf{Y})_D \tag{11}$$

Two norms can then be deduced from (10) and (11):

$$\begin{aligned}\|\mathbf{X}\|_\sigma&=\sqrt{(\mathbf{X},\mathbf{X})_\sigma}\\ \|\mathbf{X}\|_{\sigma-1}&=\sqrt{(\mathbf{X},\mathbf{X})_{\sigma-1}}\end{aligned} \tag{12}$$

It has been shown that (Prager–Synge's Theorem [13]):

$$\varepsilon^2=\|\mathbf{E}_{\text{ad}}-\sigma^{-1}\mathbf{J}_{\text{ad}}\|_\sigma^2=\|\mathbf{E}_{\text{ad}}-\mathbf{E}_{\text{ex}}\|_\sigma^2+\|\mathbf{J}_{\text{ad}}-\mathbf{J}_{\text{ex}}\|_{\sigma-1}^2=4\left\|\frac{\mathbf{E}_{\text{ad}}+\sigma^{-1}\mathbf{J}_{\text{ad}}}{2}-\mathbf{E}_{\text{ex}}\right\|_\sigma^2 \tag{13}$$

The scalar ε is the distance between the fields $(\mathbf{E}_{\text{ad}},\mathbf{J}_{\text{ad}})$ and this distance is proportional to the distance between an average field $(\mathbf{E}_{\text{ad}}+\sigma^{-1}\mathbf{J}_{\text{ad}})/2$ and the exact solution. The scalar ε is an

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error estimator and can be calculated from a pair of admissible fields $(\mathbf{E}_{ad}, \mathbf{J}_{ad})$, using (13). The dimension of ε is $\mathbf{J}^{1/2}$ (J:Joules) and has no real physical meaning. In that sense, the unit of the error will not be indicated in the following. It can be shown also that the following inequality exists:

$$\begin{aligned} W_E &= \tfrac{1}{2}\|\mathbf{E}_{ad}\|_\sigma^2 \\ W_{ex} &= \tfrac{1}{2}\|\mathbf{E}_{ex}\|_\sigma^2 = \frac{1}{2}\|\mathbf{J}_{ex}\|_{\sigma-1}^2, \quad W_J \leqslant W_{ex} \leqslant W_E \\ W_J &= \tfrac{1}{2}\|\mathbf{J}_{ad}\|_{\sigma-1}^2 \end{aligned} \tag{14}$$

It can also be seen from (14) that the admissible fields yield energies W_E and W_J that bound the exact energy W_{ex} of the system.

3.3. Calculation of the admissible solutions

In the following, we will show how the two admissible fields $\mathbf{E}_{ad}$ and $\mathbf{J}_{ad}$ can be calculated by solving two complementary problems. In fact, two potential formulations can be used to solve the equation system described in Section 3.1. As $\mathbf{E}$ is curl free:

$$\mathbf{E} = -\mathbf{grad}\,\varphi + V\,\boldsymbol{\beta} \tag{15}$$

With φ a function of $L^2(D)$ such that:

$$\varphi = 0 \text{ on } S_{E1} \text{ and } \varphi = 0 \text{ on } S_{E2} \tag{16}$$

With $\boldsymbol{\beta} \in \mathbf{L}^2_{curl}$ a curl free function with a circulation equal to one between S_{E1} and S_{E2}. $\mathbf{E}$, written under the form (15), satisfies implicitly (4) and (6) and is, therefore, admissible. The equation to be solved in the scalar potential formulation is then:

$$\operatorname{div}\sigma\,\mathbf{grad}\,\varphi = \mathbf{V}\,\operatorname{div}(\sigma\,\boldsymbol{\beta}) \tag{17}$$

Similarly, since $\mathbf{J}$ is divergence free, it derives from the curl of a vector potential $\mathbf{T} \in \mathbf{L}^2_{curl}$. Since the surface S_J is not contractible, the boundary conditions for the tangential component of the vector $\mathbf{T}$ are not homogeneous. In fact, the flux flowing across any surface in D with boundary on S_J is zero and consequently the fluxes flowing across S_{E1} and S_{E2} are zero. To account for the boundary conditions (7), an additional field $\mathbf{N} \in \mathbf{L}^2_{div}$ can be introduced such that:

$$\operatorname{div}\mathbf{N} = 0 \quad \text{and} \quad \int_{S_{E1}} \mathbf{N}\cdot\mathbf{n}\,dS = -\int_{S_{E2}} \mathbf{N}\cdot\mathbf{n}\,dS \tag{18}$$

Then, the current density can be written as:

$$\mathbf{J} = \mathbf{curl}\,\mathbf{T} + I\mathbf{N} \tag{19}$$

$$\mathbf{T}\times\mathbf{n} = 0 \quad \text{on } S_J \tag{20}$$

Let I be the current flowing from S_{E1} to S_{E2}. Introducing (19) into (4), the equation to be solved with the vector potential $\mathbf{T}$ as unknown is then:

$$\mathbf{curl}[\sigma^{-1}\mathbf{curl}\ \mathbf{T}] = -I\,\mathbf{curl}[\sigma^{-1}\mathbf{N}] \tag{21}$$

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The current I is a natural source term for the vector potential formulation but since in our problem the circulation of $\mathbf{E}$ is imposed, an additional relation has to be added linking the prescribed value V with the current, which becomes an additional degree of freedom (DOF) [15]:

$$-(\mathbf{E},\mathbf{N})=V \tag{22}$$

Whitney elements are commonly used to approximate the potentials and the fields in electromagnetics [16]. We consider a simplicial mesh M with n_0 nodes, n_1 edges, n_2 facets and n_3 elements. We consider the sequence of spaces W_0, W_1 and W_2 which are the nodal element space, the edge element space and the facet element space, respectively. As D is contractible, we have the following properties:

$$\mathrm{Im}(\mathbf{grad}\ W_0)=\mathrm{Ker}(\mathbf{curl}\ W_1) \tag{23}$$

$$\mathrm{Im}(\mathbf{curl}\ W_1)=\mathrm{Ker}(\mathrm{div}\, W_2) \tag{24a}$$

With $\mathrm{Im}(f)$ and $\mathrm{Ker}(f)$ the co-domain and the kernel of the operator f respectively. According to (15) and (23), φ and E are in W_0 and W_1 respectively. According to (19) and (24a), $\mathbf{T}$ and $\mathbf{J}$ (and N) are in W_1 and W_2, respectively. We denote w_{0i} the nodal shape function associated with node i, w_{1i} the shape function associated with edge i and w_{2i} the shape function associated with facet i. We denote N_{0E} and N_{1E} the sets of nodes and edges located on the boundary S_E. The potential φ and the field $\mathbf{E}$ are then written as:

$$\varphi=\sum_{\substack{i=1\\ i\notin N_{0E}}}^{n0}\varphi_i w_{0i} \quad \text{and} \quad \mathbf{E}=\sum_{\substack{i=1\\ i\notin N_{1E}}}^{n_1} E_i \mathbf{w}_{1i} \tag{24b}$$

The fields φ and $\mathbf{E}$ satisfy implicitly the boundary conditions (16) and (6) on S_E. Similarly, if N_{1J} and N_{2J} are the sets of edges and facets on the surfaces S_J, the potential $\mathbf{T}$ and the current density $\mathbf{J}$ can be approximated by:

$$\mathbf{T}=\sum_{\substack{i=1\\ i\notin N_{1J}}}^{n_1} T_i \mathbf{w}_{1i} \quad \text{and} \quad \mathbf{J}=\sum_{\substack{i=1\\ i\notin N_{2J}}}^{n_2} J_i \mathbf{w}_{2i} \tag{25}$$

To solve both potential formulations numerically, the weak forms of (15) or of (18) are used and the Galerkin method is applied. The solution of the two potential formulations with a numerical method gives the pair of admissible fields as ($\mathbf{E}_{\mathrm{ad}}=-\mathbf{grad}\,\varphi+V\boldsymbol{\beta}$, $\mathbf{J}_{\mathrm{ad}}=\mathbf{curl}\ \mathbf{T}+I\mathbf{N}$). The numerical error can then be estimated using (13).

3.4. Application

Consider a structure made of four aluminum sections. Three identical sections are affixed on the fourth (main section). We propose to take into account the contact resistances that exist between the upper sections and the main section when they are affixed together. To do so, an intermediate resistive layer is introduced between each two sections in contact. The geometry of the structure is shown in Figure 1. Figure 2 shows the boundary conditions. The normal component of the current density cancels everywhere except on the three surfaces where the tangential component of $\mathbf{E}$ is equal to zero. On these three surfaces, the voltage is imposed. We denote I_1 and I_2 the currents flowing through the surface on which the voltage is prescribed as 1 V. The conductivity of the

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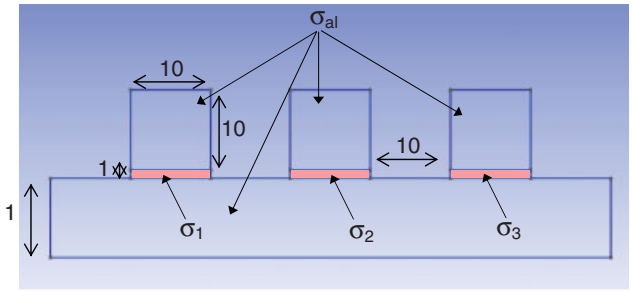

Figure 1. Description of the device studied (dimension are in mm).

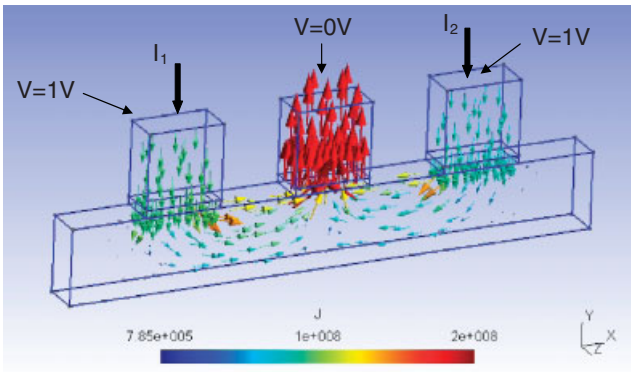

Figure 2. Boundary conditions and current density distribution calculated using the vector potential formulation.

Table I. Specifications of the four meshes.

	Mesh 1	Mesh 2	Mesh 3	Mesh 4
Number of nodes	148	352	918	2747
Number of edges	675	1733	4873	15421
Number of unknowns φ	131	324	870	2647
Number of unknowns $\mathbf{T}$	331	882	2771	9599

aluminum is equal to $37\,\mathrm{MS\,m^{-1}}$, whereas in the contact resistance area the conductivity is equal to $0.64\,\mathrm{MS\,m^{-1}}$. The current density distribution is given in Figure 2. We have considered four meshes whose specifications are given in Table I. Meshes 1 and 2 are shown in Figure 3.

The initial problem was solved with the four meshes and the two potential formulations. The scalar potential φ was expanded using linear nodal shape functions (24b). The vector potential $\mathbf{T}$ was decomposed in the edge element space (25). An estimate of the numerical error was calculated from φ and $\mathbf{T}$, using (13). Figure 4 shows the evolution of the error estimate reported as a function of the number of elements on a log–log scale. The error decreases almost linearly with the increase

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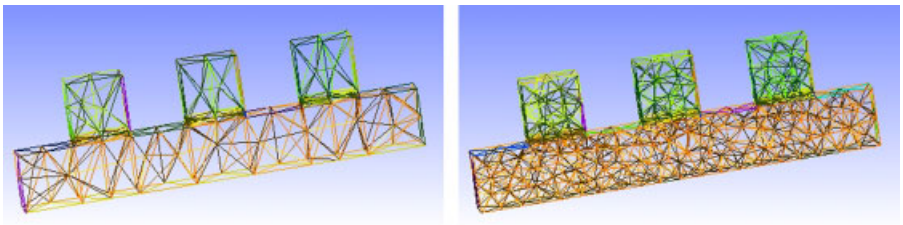

Figure 3. Mesh 1 (left) and Mesh 2 (right).

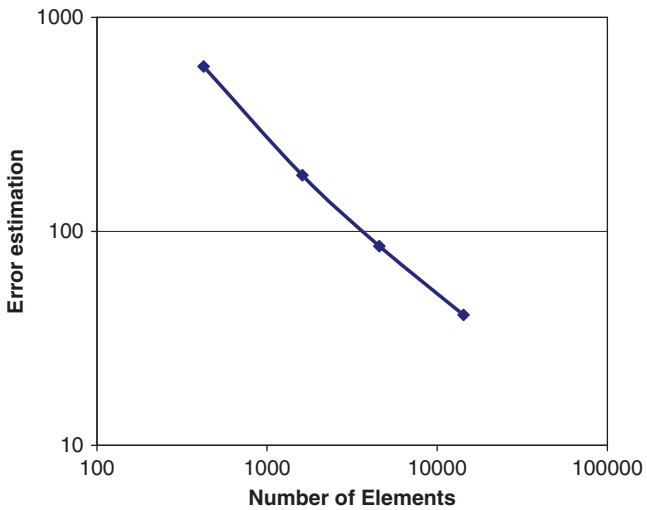

Figure 4. Evolution of the error estimate as a function of the number of elements of the mesh on a log–log scale.

in the number of elements. This behavior is expected from the fact that linear functions were used for the approximation. The evolution of the energies given for the two formulations is given in Figure 5. Both energies converge to the exact solution bounding it, as the number of elements increases, as was indicated in (14).

4. STOCHASTIC ELECTROKINETIC PROBLEM

We now assume that some conductivities σ_i are random variables. The conductivity is then written as:

$$\sigma(x,\theta)=\sum_{i=1}^{N}\sigma_i(\theta)I_i(x) \tag{26}$$

With the functions $I_i(x)$ defined in (9).

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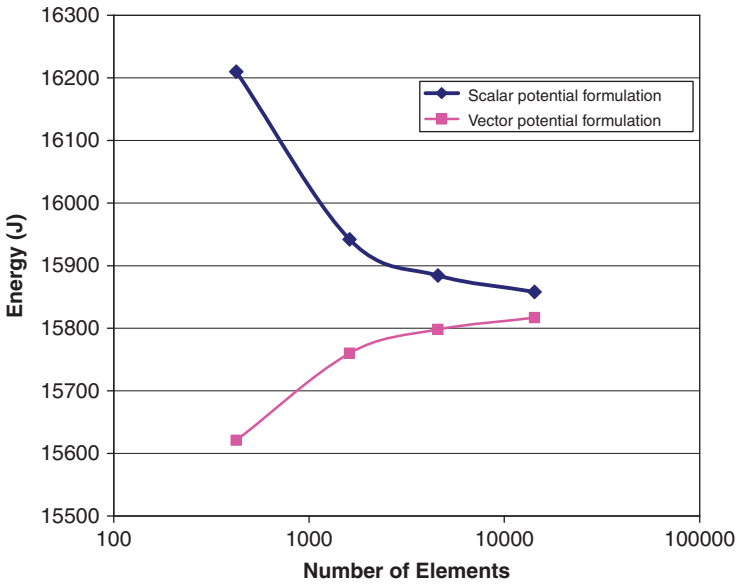

Figure 5. Evolution of the energies given by the two formulations as a function of the number of elements.

Remark

In the literature, it is often assumed that the coefficient of the constitutive law (for example the conductivity in our case) is a random field with a given covariance function. The random field is then approximated by a truncated Karhunen–Loeve expansion [17]. The expression is very similar to the expression in (26) where the support of the functions $I_i(x)$ depending on the x parameter is defined on the whole domain and where the random variables $\sigma_i(\theta)$ are uncorrelated. For the sake of simplicity, they are often assumed to be independent. Consequently, the following can be easily extended to the case where a truncated Karhunen–Loeve expansion is used to represent the conductivity with independent random variables.

If the conductivity σ_i has a constant value (i.e. deterministic) on a subdomain D_i, it can be still considered a random variable $\sigma_i(\theta)$ with a single value σ_I with a probability equal to one. Therefore, (19) holds even if conductivities σ_i are constant on subdomains D_i. From (9), it can be seen that the electric field intensity $\mathbf{E}$ and the current density $\mathbf{J}$ are now also random and depend on $\theta \in \Omega$. We will assume that the boundaries of the domain and the boundary conditions are known exactly. The boundary conditions are the same as those of the deterministic problem. The problem to be solved is given by Equations (4)–(7) and (26). The conductivities are assumed to belong to a bounded strictly positive interval that reads:

$$0 < \sigma_i^{\min} \leqslant \sigma_i(\theta) \leqslant \sigma_i^{\max} \quad i \in [1, N] \tag{27}$$

For each combination of the N-tuple $(\sigma_1, \ldots \sigma_N)$, the random conductivities are bounded according to (27). The problem is well posed and has a unique solution denoted as $(\mathbf{E}_{\text{ex}}(x,\ \theta), \mathbf{J}_{\text{ex}}(x, \theta))$ [17]. It is useful, as we will see later, to define the space of approximation such that the input data of the problem depend on classical random variables such as standard normal random

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variables, uniform random variables, etc. Therefore, in the following, we will assume that $\sigma_i(\theta)$ is a function of a random variable $\xi_i(\theta)$ defined on the interval Γ_i with a probability density function $\rho_i(\xi_i)$. We denote $\Gamma = \bigotimes_{i=1}^{N} \Gamma_i$ and the joint probability density function $\rho(\boldsymbol{\xi}) = \rho(\xi_1, \xi_2, \ldots, \xi_n) = \prod_{i=1}^{N} \rho_i(\xi_i)$.

Remark
If the N random variables $\sigma_1(\theta), \ldots, \sigma_N(\theta)$ are not independent of each other, it may be possible to express them as a function of independent random variables ξ_i using appropriate transformations.

If we consider now that two fields are admissible if we have for any θ of Ω:

$$\mathbf{curl}\ \mathbf{E}(x,\theta) = \mathbf{0}, \quad \mathbf{E}(\mathrm{x},\theta) \times \mathbf{n} = \mathbf{0} \quad \text{on } S_{\mathrm{E}} \tag{28}$$

$$\mathrm{div}\,\mathbf{J}(x,\theta) = \mathbf{0}, \quad \mathbf{J}(\mathrm{x},\theta) \cdot \mathbf{n} = \mathbf{0} \quad \text{on } S_J \tag{29}$$

We get for any $\theta, \mathbf{E}(x,\theta)$ and $\mathbf{J}(x,\theta)$ that satisfy (13).

We consider the expectation $\langle \varepsilon^2 \rangle = E[\varepsilon^2(\theta)] = E[\|\mathbf{E}(x,\theta) - \sigma^{-1}(x,\theta)\mathbf{J}(x,\theta)\|_\sigma^2]$. The term is equal to zero if $\mathbf{E}(x,\theta)$ and $\mathbf{J}(x,\theta)$ satisfy the constitutive relations in a mean-square sense and so the couples $(\mathbf{E}(x,\theta), \mathbf{J}(\mathrm{x},\theta))$ and $(\mathbf{E}_{\mathrm{ex}}(x,\theta), \mathbf{J}_{\mathrm{ex}}(x,\theta))$ are equal in a mean-square sense. Moreover, according to (13), ε is a measure of the distance between the exact solution and an average field $\mathbf{E}_{avg}(x,\theta)$:

$$\langle \varepsilon^2 \rangle = 4\, E[\|\mathbf{E}_{\mathrm{avg}}(x,\theta) - \mathbf{E}_{\mathrm{ex}}(x,\theta)\|_\sigma^2] \quad \text{with } \mathbf{E}_{\mathrm{avg}}(x,\theta) = \frac{\mathbf{E}(x,\theta) + \sigma^{-1}(x,\theta)\mathbf{J}(x,\theta)}{2} \tag{30}$$

We also define the energies $W_E(x,\theta)$ and $W_J(x,\theta)$ (see (14)):

$$\begin{aligned} W_E(\theta) &= \tfrac{1}{2}\|\mathbf{E}(x,\theta)\|_\sigma^2 \\ W_{\mathrm{ex}}(\theta) &= \tfrac{1}{2}\|\mathbf{E}_{\mathrm{ex}}(x,\theta)\|_\sigma^2 = \tfrac{1}{2}\|\mathbf{J}_{\mathrm{ex}}(x,\theta)\|_{\sigma^{-1}}^2 \\ W_J(\theta) &= \tfrac{1}{2}\|\mathbf{J}(x,\theta)\|_{\sigma^{-1}}^2 \end{aligned} \tag{31}$$

Using an approach similar to the one proposed in [12], we look at the cumulative probability density function $\Pi_E(W)$, $\Pi_{\mathrm{ex}}(W)$ and $\Pi_J(W)$:

$$\begin{aligned} \Pi_E(W) &= P(W_E(\theta) \leqslant W) \\ \Pi_{\mathrm{ex}}(W) &= P(W_{\mathrm{ex}}(\theta) \leqslant W) \\ \Pi_J(W) &= P(W_J(\theta) \leqslant W) \end{aligned} \tag{32}$$

Then, according to the inequality (12), we have:

$$\Pi_E(W) \leqslant \Pi_{\mathrm{ex}}(W) \leqslant \Pi_J(W) \tag{33}$$

The expression above can be used to assess the accuracy of the model around the tail of the probability density function. In fact, if in that area the difference between $\Pi_E(W)$ and $\Pi_J(W)$ is high, the results given by the model cannot be considered as accurate especially if the aim is failure analysis.

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Expanding the expression of $\varepsilon(\theta)$, we obtain:

$$\varepsilon^2(\theta) = \|\mathbf{E}(x,\theta) - \sigma^{-1}(x,\theta)\mathbf{J}(x,\theta)\|_\sigma^2 = 2W_E(\theta) + 2W_J(\theta) - 2(\mathbf{E}(x,\theta), \mathbf{J}(x,\theta))_D \tag{34}$$

This expression is in fact the expression that has to be implemented to calculate the error and its expectation. The main interest in this expression is that there are at most two successive products of random fields (for example $\sigma(x,\theta)\mathbf{E}(x,\theta)\cdot\mathbf{E}(x,\theta)$) which simplify the calculation compared with the original expression (left-hand side of (34)) which has three successive products.

In the following we take advantage of these properties to estimate *a posteriori* the numerical error due to the discretization of the spatial and the random space. We are looking at solutions $\mathbf{E}(x,\theta)$ and $\mathbf{J}(x,\theta)$ in $L^2(\Omega)\otimes\mathbf{L}^2_{\text{curl}}(D)$ and $L^2(\Omega)\otimes\mathbf{L}^2_{\text{div}}(D)$. To approximate the spaces $L^2_{\text{grad}}(D)$, $\mathbf{L}^2_{\text{curl}}(D)$ and $\mathbf{L}^2_{\text{div}}(D)$, we use Whitney elements (see (23) and (24a)). To approximate the space $L^2(\Omega)$, we use a subset of orthogonal functions $(\Psi^k[\boldsymbol{\xi}(\theta)])_{1\leqslant k\leqslant P_{\text{out}}}$ of a basis $(\Psi^k[\boldsymbol{\xi}(\theta)])_{1\leqslant k\leqslant\infty}$ that spans $L^2(\Omega)$:

$$E(\Psi^i\Psi^j) = q^i\,\delta_{ij} \quad \text{and} \quad E(\Psi^i) = m^i \tag{35}$$

Where (m^i, q^i) are real numbers and δ_{ij} the Kronecker delta function. Generally, the functions $(\Psi^k[\xi(\theta)])_{1\leqslant k\leqslant P_{\text{out}}}$ are normalized and the q^i are equal to one.

Different basis functions can be used such as the Wiener–Askey polynomials [6, 18, 19] or the Wiener–Haar decomposition [5]. The 'best' choice for the orthogonal basis for the approximations is always an issue. In [5], it is shown that the Wiener–Haar expansion is more robust when the solution has a steep dependence (or discontinuity) on the input random variables of the problem but when the solution is smooth, the Wiener–Askey expansion exhibits faster convergence. In practice it is convenient to use the same basis of approximation in the random dimension to solve both potential formulations. We will assume in the following that we have two fields $\mathbf{E}(x,\theta)$ and $\mathbf{J}(x,\theta)$. These fields are given, after solving the initial stochastic model numerically and are written as:

$$\begin{aligned}
\mathbf{E}(x,\theta) &= \sum_{i=1}^{n_1}\sum_{j=1}^{P_{\text{out}}} \mathbf{E}_i^j w_{1i}(x)\Psi^j(\theta) = \sum_{i=1}^{n_1}\mathbf{E}_i(\theta)w_{1i}(x) = \sum_{j=1}^{P_{\text{out}}}\mathbf{E}^j(x)\Psi^j(\theta)\\
\mathbf{J}(x,\theta) &= \sum_{i=1}^{n_2}\sum_{j=1}^{P_{\text{out}}} \mathbf{J}_i^j w_{2i}(x)\Psi^j(\theta) = \sum_{i=1}^{n_2}\mathbf{J}_i(\theta)w_{2i}(x) = \sum_{j=1}^{P_{\text{out}}}\mathbf{J}^j(x)\Psi^j(\theta)
\end{aligned} \tag{36}$$

As $(\Psi^j)_{1\leqslant j\leqslant P_{\text{out}}}$ are linearly independent functions, it means that for all j $(1\leqslant j\leqslant P_{\text{out}})$, $\mathbf{E}^j(x)$ and $\mathbf{J}^j(x)$ are admissible in the 'deterministic' sense (i.e. $\mathbf{E}^j(x)$ satisfying (4) and (6) and $\mathbf{J}^j(x)$ satisfying (5) and (7)). To calculate the error ε, we also need to expand the conductivities σ_i and their inverse:

$$\begin{aligned}
\sigma_i(\theta) &= \sum_{j=1}^{\infty}\sigma_i^j\Psi^j(\theta)\\
\sigma_i^{-1}(\theta) &= \sum_{j=1}^{\infty}\text{inv}\sigma_i^j\Psi^j(\theta)
\end{aligned} \tag{37}$$

Generally, the expansion can be finite or infinite. Depending on the probability density function of the conductivity and of the probability density functions of the ξ_i. For example, if a conductivity σ_i is a uniform random variable, the Wiener–Askey expansion with Legendre polynomial ξ_i is finite,

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having only two terms. The Wiener–Askey expansion with Hermite polynomial (see Section 6 for more details) is infinite. The expressions of the previous terms in (31) are:

$$(\mathbf{E}(x,\theta),\mathbf{J}(x,\theta))_D = \sum_{i=1}^{P_{\text{out}}}\sum_{j=1}^{P_{\text{out}}} (\mathbf{E}^i(x),\mathbf{J}^j(x))_D \Psi^i(\theta)\Psi^j(\theta)$$

$$W_E(\theta) = \frac{1}{2}\sum_{i=1}^{P_{\text{out}}}\sum_{j=1}^{P_{\text{out}}}\sum_{m=1}^{\infty}\sum_{l=1}^{N} \sigma_l^m (I_l(x)\mathbf{E}^i(x),\mathbf{E}^j(x))_D \Psi^i(\theta)\Psi^j(\theta)\Psi^m(\theta) \tag{38}$$

$$W_J(\theta) = \frac{1}{2}\sum_{i=1}^{P_{\text{out}}}\sum_{j=1}^{P_{\text{out}}}\sum_{m=1}^{\infty}\sum_{l=1}^{\mathrm{N}} {}_{\text{inv}}\sigma_l^m (I_l(x)\mathbf{J}^i(x),\mathbf{J}^j(x))_D \Psi^i(\theta)\Psi^j(\theta)\Psi^m(\theta)$$

As $I_l(x)$ is equal to 1 in D_l and zero elsewhere, the scalar product has to be calculated only on D_l which can reduce the computation effort. Now, we write the three terms as a linear combination of $(\Psi^k)_{1\leqslant k\leqslant\infty}$. The product $(\Psi^j\Psi^i)_{1\leqslant i\leqslant P_{\text{out}},\ 1\leqslant j\leqslant P_{\text{out}}}$ can be written as a linear combination of $(\Psi^k)_{1\leqslant k\leqslant\infty}$:

$$\Psi^i(\theta)\Psi^j(\theta) = \sum_{k=1}^{\infty} d^{ijk}\Psi^k(\theta) \tag{39}$$

As the basis has been assumed orthogonal it is easy to see that the coefficient d^{ijk} is given by:

$$d^{ijk} = \frac{E(\Psi^i(\theta)\Psi^j(\theta)\Psi^k(\theta))}{E(\Psi^k(\theta)^2)} \tag{40}$$

If the Wiener–Askey expansion is used, then Ψ^j is an N-dimensional polynomial of random variables of maximum order p_{out} that yields $P_{\text{out}} = C_{N+p_{\text{out}}}^{p_{\text{out}}}$ basis functions. The product $\Psi^i\Psi^j$ is also an N-dimensional polynomial (N is the number of random variables, see (26)) with an order of at most $2^* p_{\text{out}}$. This product can be written as a linear combination of the basis functions Ψ^k with $1\leqslant k\leqslant {}^2P_{\text{out}}$ (${}^2P_{\text{out}}$ is the number of the basis functions with order less or equal to $2^* p_{\text{out}}$) that are orthogonal by definition to all basis function Ψ^k with $k > {}^2P_{\text{out}}$ Then, the expectation $\Psi^i\Psi^j\Psi^k$ is equal to zero, as is the term d^{ijk}, if the index k is higher than that ${}^2P_{\text{out}}$ (the order of Ψ^k is greater than $2^* p_{\text{out}}$). Consequently, the sum in that case is finite (39) and we have:

$$\Psi^i(\theta)\Psi^j(\theta) = \sum_{k=1}^{{}^2P_{\text{out}}} d^{ijk}\Psi^k(\theta) \tag{41}$$

If the Wiener–Haar decomposition is used, the space of approximation is stable with the product meaning that the product of two basis functions $\Psi^k\Psi^j$ can be always written has a linear combination of the basis functions $(\Psi^j)_{1\leqslant i\leqslant P_{\text{out}}}$. Consequently, the sum is also finite and the number of terms is ${}^2P_{\text{out}} = P_{\text{out}}$. In the following, we will assume that the sum is finite and equals to ${}^2P_{\text{out}}$. If, for a particular numerical implementation, this condition is not satisfied, a *de facto* truncation

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will be necessary until this condition is fulfilled

$$(\mathbf{E}(x,\theta),\mathbf{J}(x,\theta))_D = \sum_{k=1}^{2P_{\text{out}}} a^k \Psi^k(\theta), \quad a^k = \sum_{i=1}^{P_{\text{out}}}\sum_{j=1}^{P_{\text{out}}} d^{ijk}(\mathbf{E}^i(x),\mathbf{J}^j(x))_D$$

$$W_E(\theta) = \frac{1}{2}\sum_{l=1}^{N}\sum_{m=1}^{\infty}\sigma_l^m \sum_{k=1}^{2P_{\text{out}}} b^{kl}\Psi^k(\theta)\Psi^m(\theta)$$

$$b^{kl} = \sum_{i=1}^{P_{\text{out}}}\sum_{j=1}^{P_{\text{out}}} d^{ijk}(I_l(x)\mathbf{E}^i(x),\mathbf{E}^j(x))_D \tag{42}$$

$$W_J(\theta) = \frac{1}{2}\sum_{l=1}^{N}\sum_{m=1}^{\infty}{}_{\text{inv}}\sigma_l^m \sum_{k=1}^{2P_{\text{out}}} c^{kl}\Psi^k(\theta)\Psi^m(\theta)$$

$$c^{kl} = \sum_{i=1}^{P_{\text{out}}}\sum_{j=1}^{P_{\text{out}}} d^{ijk}(I_l(x)\mathbf{J}^i(x),\mathbf{J}^j(x))_D$$

As the basis is orthogonal and using the expression (34), we have:

$$\langle\varepsilon^2\rangle = E(\varepsilon^2(\theta)) = \sum_{l=1}^{N}\left[\sum_{k=1}^{2P_{\text{out}}} q^k[\sigma_l^k b^{kl} + {}_{\text{inv}}\sigma_l^k c^{kl}] - 2\sum_{k=1}^{2P_{\text{out}}} m^k a^k\right] \tag{43}$$

First, we note that the second sum is generally reduced to one term a_1 because the expectation of the basis function generally vanishes except for the first term of the basis. Second, only the $2P_{\text{out}}$ terms of the conductivities and their inverses are involved in the expression of the error. Consequently, to calculate the error numerically, we need only the truncated expansion of the conductivities up to $2P_{\text{out}}$. The other terms are not required.

Remark

The term $\langle\varepsilon\rangle$ gives an information on the error on the whole expression of $\mathbf{E}(x,\theta)$ and $\mathbf{J}(x,\theta)$ given by (36). It is difficult to extract from the relationship (43) information to gain insight into the error on each statistical moment of the fields. In fact, the moment of order nth of the fields is the expectation of the term $\mathbf{E}(x,\theta)-\mathbf{E}_0(x)$ (or $\mathbf{J}(x,\theta)-\mathbf{J}_0(x)$) raised to the power of n and $\langle\varepsilon\rangle$ depends on the expectation of the term $\mathbf{E}(x,\theta)-\sigma(x,\theta)\ \mathbf{J}(x,\theta)$. Nevertheless, if we have look at the energy, we have $W_J(\theta)\leqslant W_{\text{ex}}(\theta)\leqslant W_E(\theta)$, we can deduce some inequalities on the statistical moments on the energy. If we denote m_J, m_{ex} and m_E the means of $W_J(\theta)$, $W_{\text{ex}}(\theta)$ and $W_E(\theta)$, respectively, we have $m_J\leqslant m_{\text{ex}}\leqslant m_E$. Moreover, since the energy is always positive, the variance of the energy $V[W_{\text{ex}}(\theta)]$ is bound. In fact, we have: $E[W_J^2(\theta)]-m_E\leqslant V[W_{\text{ex}}(\theta)]\leqslant E[W_E^2(\theta)]-m_J$.

5. APPLICATION

5.1. Description of the problem

Consider again the structure in Figure 1, described so far in a deterministic context. Here we will assume that the contact resistances are random. The conductivity of the four aluminum sections are assumed to be known and fixed at $\sigma_{\text{al}}=33\,\text{MS}\,\text{m}^{-1}$. The three thin layers between the four aluminum sections can have uniform but random conductivities. We will consider two

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cases. In the first case, the conductivity $\sigma_1(\theta)$ is assumed to be a uniform random variable whereas conductivities σ_2 and σ_3 are constant. In the second case, the three conductivities are independent uniform random variables. In the following, we will develop the calculation only for the second case, the first case can be easily deduced from the first. To approximate the space $L^2(\Omega)$, we use the Wiener–Askey polynomial expansion. In [18], it has been shown that the choice of polynomial basis plays an important role in the convergence rate towards the exact solution. We will try to show the same behavior by using the error estimator. We will consider two different polynomial expansions based on the Legendre polynomials $(\Psi_p^k[u_1(\theta), u_2(\theta), u_3(\theta)])_{1\leqslant k\leqslant P_{\text{out}}}$ with $u_1(\theta), u_2(\theta)$, and $u_3(\theta)$ independent uniform random variables in the interval $[-1,1]$ and the Hermite polynomials $(\Psi_H^k[\xi_1(\theta), \xi_2(\theta), \xi_3(\theta)])_{1\leqslant k\leqslant P_{\text{out}}}$ with $\xi_1(\theta), \xi_2(\theta)$ and $\xi_3(\theta)$ independent standard normal random variables.

To calculate the expansion of the admissible electric field $\mathbf{E}(x,\theta)$ on a mesh M, we use the scalar potential formulation. Using the orthogonality of the approximation basis, the term $\mathbf{E}_i^j$ of the expansion introduced in (36) can be calculated as:

$$\mathbf{E}_i^j = \frac{E[\mathbf{E}_i(\theta)\Psi^j(\theta)]}{E[\Psi^j(\theta)^2]} \tag{44}$$

The denominator can generally be calculated analytically. The numerator can be estimated using a Monte Carlo Simulation or other sampling methods. The experience in the literature [20] indicates that for a small number of input random variables, it is more suitable to use a quadrature method to approximate the integral:

$$E[\mathbf{E}_i(\theta)\Psi^j(\theta)] = \int_\Gamma \mathbf{E}_i(\boldsymbol{\xi})\Psi^j(\boldsymbol{\xi})\rho(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} \tag{45}$$

In our case, the Gauss quadrature was used but it is worth mentioning that other methods, such as sparse grid methods (Smolyack's quadrature) can be applied equally well. In that case the previous term is approximated by the following expression:

$$E[\mathbf{E}_i(\theta)\Psi^j(\theta)] = \sum_{i_1=1}^{\mathrm{m}_1} \dots \sum_{i_n=1}^{m_n} \omega_{i1}\dots\omega_{\text{in}}\mathbf{E}_j(\xi_{i1},\dots,\xi_{\text{in}})\Psi^j(\xi_{i1},\dots,\xi_{\text{in}}) \tag{46}$$

The choice of the Gauss points $(\xi_{ij})_{1\leqslant ij\leqslant mj}$ and of the associated weights $(\omega_{ij})_{1\leqslant ij\leqslant mj}$ depends on the probability density function associated with the jth input random variable $\xi_j(\theta)$. If the input random variable $\xi_j(\theta)$ is standard normal then the Gauss points will be the roots of the Hermite polynomials of order m_j. In the same way, if the input random variable $\xi_j(\theta)$ is uniform then the Gauss points will be the roots of the Legendre polynomials of order m_j. It should be noted that since each term $\mathbf{E}_j(\xi_{i1},\dots,\xi_{\text{in}})$ is obtained from the solution of a scalar potential formulation, the random field $\mathbf{E}(x,\theta)$ given by (36) remains admissible even though the terms $\mathbf{E}_i^j$ are an approximation of (44). We note as well that the error estimator will also take into account the discrepancy error introduced by the quadrature method in addition to the error due to the approximation in both the spatial and random dimensions.

Remark

We mentioned above that the numerator of (44) could also be estimated using a sampling method. In that case too, the estimation error will be taken into account in the term $\langle\varepsilon\rangle$ given by (46). It should be noted that the error estimator can be used here to determine the number of sampling

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points *online*. In fact, the error $\langle\varepsilon\rangle$ can be calculated during the sampling process which can be stopped as soon as the variation of $\langle\varepsilon\rangle$ remains small (that is to say that the sampling error on the term E_i^j is negligible versus the errors of approximation).

5.2. Error calculation

First, we consider the case in which only one conductivity (σ_1) is a uniform random variable on the interval $[\sigma_{\min}, \sigma_{\max}]$. The values $\sigma_{\min}$ and $\sigma_{\max}$ are reported in Table II and also the mean and the standard deviation. The extreme values of $\sigma_{\min}$ and $\sigma_{\max}$ correspond approximately to the variation of the conductivities that may be present when two aluminium parts are welded together [21]. For each mesh, the expansion up to the order $p_{\text{out}}=6$ for the electric field intensity and for the current density were calculated using the projection method using both Hermite (ξ_1 is a standard normal variable) and Legendre (ξ_1 is uniform between $[-1,1]$) to approximate the random dimension. The conductivity and its inverse must also be expanded using these polynomials. The decomposition of the conductivity using Legendre polynomials is straightforward because only two terms are needed; σ_1^1—the average of the conductivity and σ_1^2—half of the length of the interval of variation of the random variable. The expansion of the inverse has an infinite number of terms. To calculate them we used a Galerkin method as proposed in [22]. The expansion of the conductivity σ_1^{-1} was truncated up to the order $p_{\text{in}}=12$ and we denote σ_{1p}^{-1} as its approximation. The terms $({}_{\text{inv}}\sigma_{1p}^j)_{1\leqslant j\leqslant p_{\text{in}}}$ were calculated as follows. The product $\sigma_1\sigma_{1p}^{-1}$ is projected on the base $(\Psi^k)_{1\leqslant k\leqslant P_{\text{in}}}$ (P_{in} is the number of terms of the polynomial basis corresponding to an order equal to p_{in}). Using the terms d^{ijk} introduced in (40), we obtain the following linear system to be solved to get the values of $({}_{\text{inv}}\sigma_l^i)_{1\leqslant i\leqslant P_{\text{in}}}$:

$$\sum_{i=1}^{P_{\text{in}}}\left(\sum_{j=1}^{P_{\text{in}}} d^{ijk}\sigma_1^j\right){}_{\text{inv}}\sigma_1^i=\delta_k,\quad 1\leqslant k\leqslant P_{\text{in}},\quad \delta_k=1\quad\text{if } k=1 \text{ else } k=0 \tag{47}$$

Decomposition of the conductivity using the Hermite polynomial was also used. The coefficients of the conductivity can be obtained analytically:

$$\begin{aligned}\sigma_1^1&=\frac{\sigma_{\min}+\sigma_{\max}}{2}\\ \sigma_1^{2i}&=(-1)^{i-1}\frac{\sigma_{\max}-\sigma_{\min}}{2^{2i-1}\sqrt{\pi}(2i-1)(i-1)!}\end{aligned} \tag{48}$$

The coefficients of the expansion of the inverse of the conductivity were calculated using the Galerkin approach presented in (47).

For each Gauss point $(\xi_{i1},\xi_{i2,\ldots,}\xi_{\text{in}})$ (see (46)), the conductivities $(\sigma_{1,\ldots,}\sigma_n)$ and their inverse $(\sigma_{1,\ldots,}^{-1}\sigma_n^{-1})$ are calculated using (37). The scalar and vector potential formulations are solved and the quadrature vectors $\mathbf{E}(\xi_{i1},\xi_{i2,\ldots,}\xi_{\text{in}})$ and $\mathbf{J}(\xi_{i1},\xi_{i2,\ldots,}\xi_{\text{in}})$ are used to calculate an approximation

Table II. Specifications of the uniform random law of the random conductivities.

	$\sigma_{\min}$	$\sigma_{\max}$	Mean	Standard deviation
Values (MS m^{-1})	0.141	1.13	0.635	0.286

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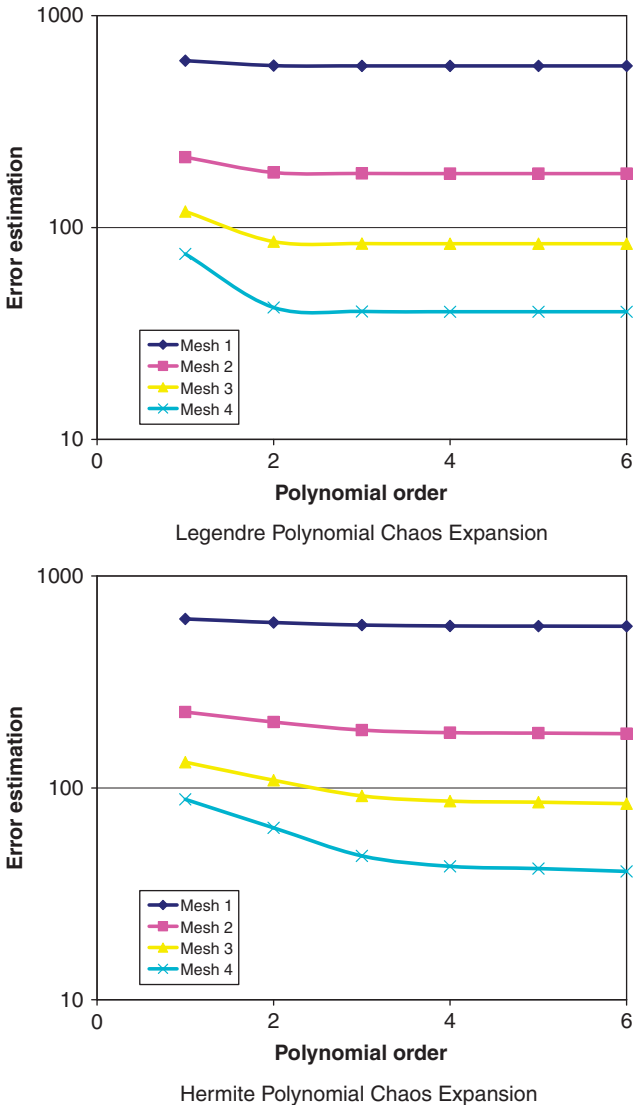

Figure 6. Evolution of the error as a function of the order of the expansion with one random conductivity.

of the terms $\mathbf{E}_i^j$ and $\mathbf{J}_i^j$. We choose the same number d of Gauss points along each 'random dimension' ξ_i with $1 \leqslant i \leqslant n$. Therefore, we have to solve the two complementary formulations d^n times to finally obtain the admissible approximated fields.

Figure 6 shows the evolution of the error versus the order of the expansion p_{out} for the 4 meshes. In other words, the error shown is calculated using (43) with the expansion up to order 2 of the fields $\mathbf{E}(x, \theta)$ and $\mathbf{J}(x, \theta)$. It is noted that the convergence is little faster with the Legendre polynomials but in both cases, for orders 3 or higher, the asymptotic value corresponding to the average of the numerical error is almost entirely due to spatial discretization. The evolution of

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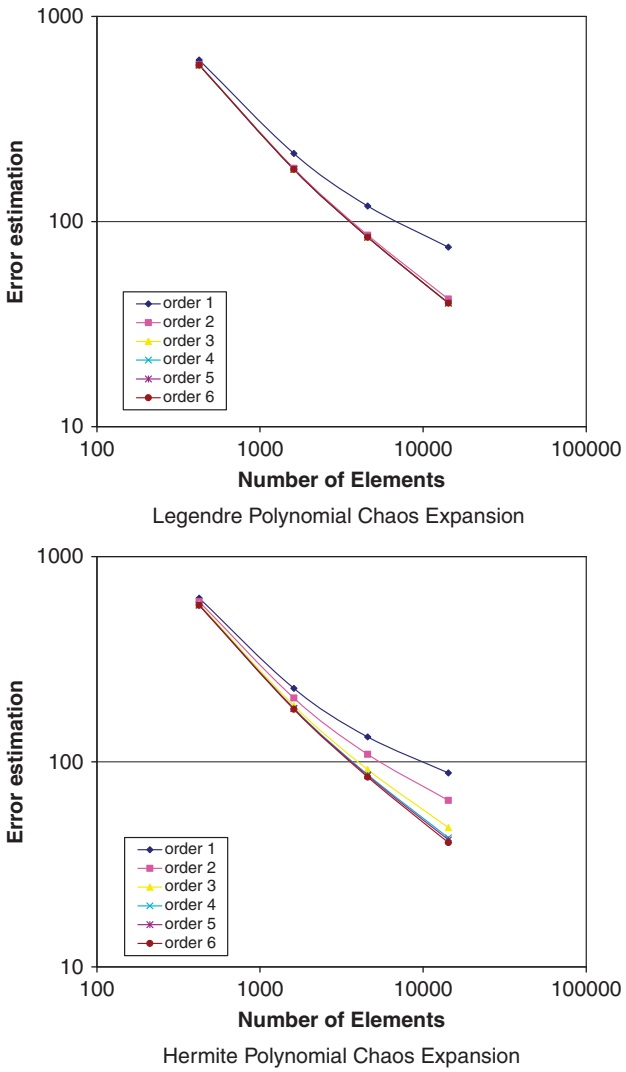

Figure 7. Evolution of the error as a function of the number of elements for different orders when only one conductivity is random.

the error as a function of the number of elements is given in Figure 7 for orders of expansion from 1 to 6. We can see that for Legendre polynomials, above the order 3, the evolution of the error is almost the same. Except for low orders, we notice that the evolution of the error is very similar to the one given in Figure 4, obtained for the deterministic case. For low orders the speed of convergence with the number of elements is slower. We can also see that increasing the order does not speed up the convergence above a given order. Clearly, a compromise has to be found.

We consider now the case of all three conductivities corresponding to the region of contact resistances being random. The evolution of the error as a function of the expansion order is given in

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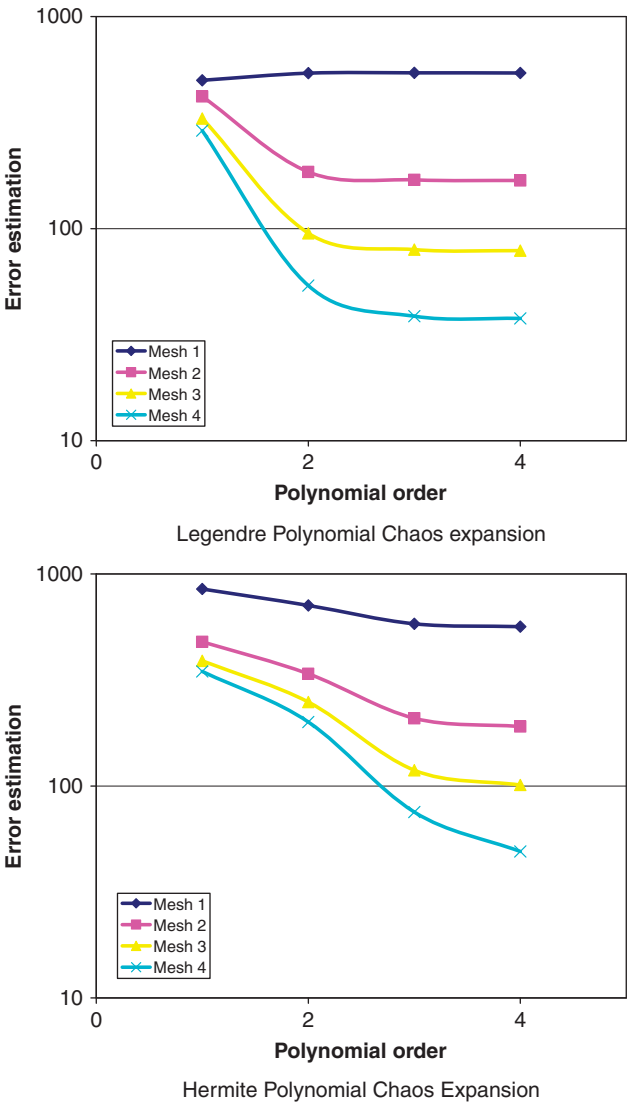

Figure 8. Evolution of the error as a function of the order of the expansion with 3 random conductivities.

Figure 8. A faster convergence with the Legendre Polynomials is noted. This appears more clearly if we look at both curves corresponding to mesh 3. In fact, with the Legendre polynomials, with order 3 the global error is almost equal to the error due to the space discretization (the error equals almost 80). With Hermite polynomials, convergence is not reached for an order 4 (i.e. the error is mostly due to space discretization). The value of the error is greater than 100. As can be seen in this example, the choice of the orthogonal polynomial basis influences the convergence speed. The evolution of the error as a function of the number of elements for a given order of expansion is shown in Figure 9. The speed of convergence depends on the order of expansion much more

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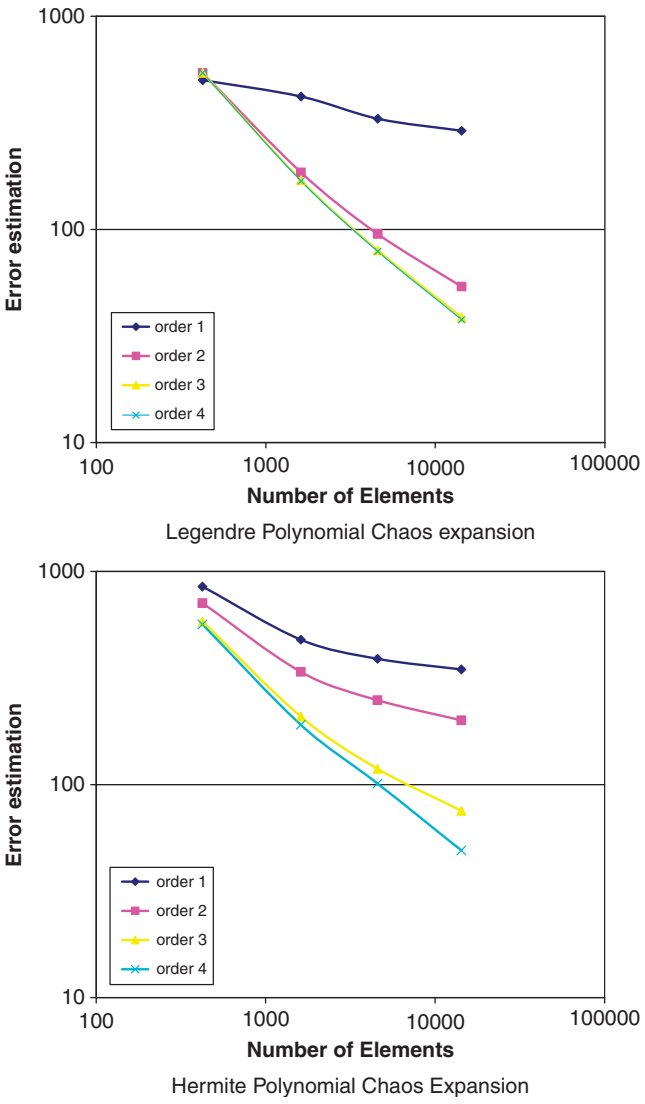

Figure 9. Evolution of the error as a function of the number of elements for different orders with 3 random conductivities.

than in the previous example. Finally, Figure 10 compares the evolution of the error as a function of the number of DoFs for both expansions. The number of DoFs is equal to the product of the number of nodes multiplied by P_{out}. The integer P_{out} depends on the order of the expansion and is equal to 4, 10, 20 and 35 for an order of expansion of 1, 2, 3 and 4, respectively. The number of DoFs is an indication of the computation requirements in terms of time and memory space. This figure confirms that for a given number of DoFs better results can be obtained with the Legendre polynomial. Nevertheless, the difference between the two expansions for a given number of DoFs

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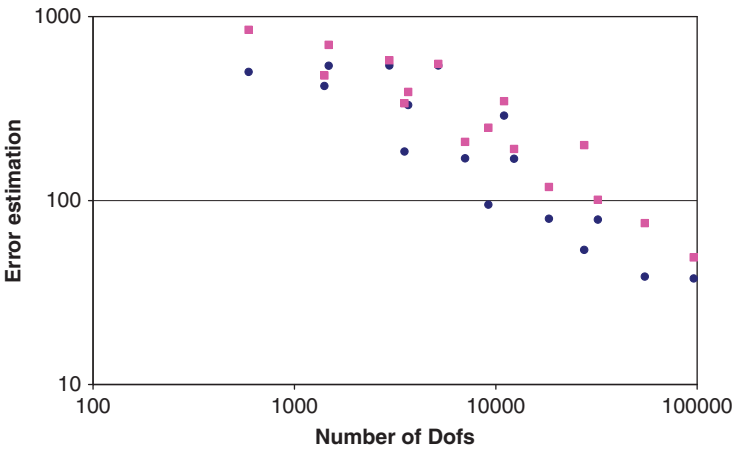

Figure 10. Evolution of the error as a function of the degrees of freedom (product of the number of nodes and P_{out}) for an expansion with multivariate Hermite Polynomials (squares) and Legendre Polynomials (circles).

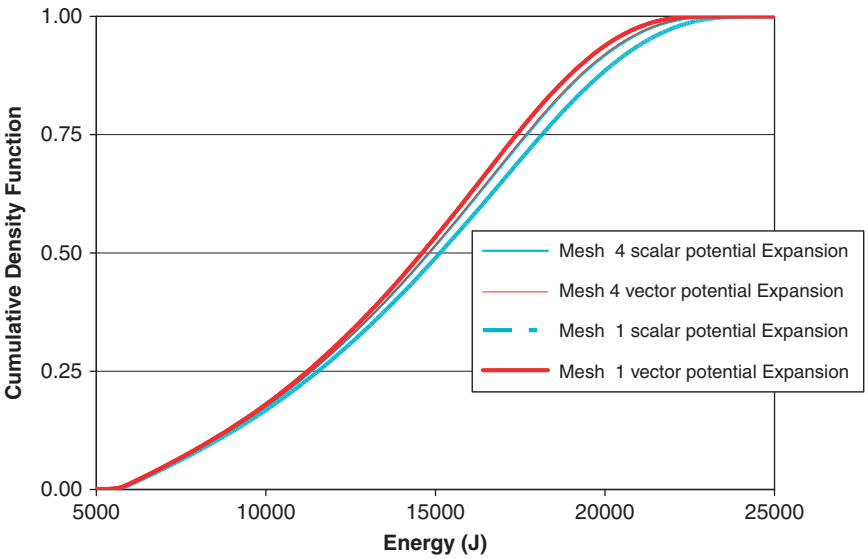

Figure 11. Evolution of the CDF of the energies calculated with the two potential formulations for two different meshes.

can be very small. As was stated in the previous example, this figure shows very clearly that the speed of convergence depends highly on the order of expansion and should be chosen carefully. The error estimator proposed here is a very useful tool for that purpose.

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Figure 11 shows the Cumulative Density Function (CDF) of the energies $W_E(\theta)$ and $W_J(\theta)$ given by (42). It shows that the property (33) is satisfied but also that for the mesh 4, the difference between the CDF obtained with the two formulations is very small.

6. CONCLUSIONS

In this paper, we studied an error estimator based on the constitutive relation error in the case of stochastic magnetostatic problems. This estimator requires two admissible fields that satisfy the equilibrium equations of the problem. We have shown how to calculate this error from a pair of admissible fields given by an expansion of orthogonal functions. The error estimate was applied to an example where both admissible fields are obtained by projection of solutions of two potential formulations. The example shows clearly the influence of the choice of the basis for the discretization of the random dimension. Moreover, the example confirms that the order of expansion of the basis must be chosen according to the spatial mesh and the error estimator is an efficient tool for that purpose. This can be eventually incorporated in an automatic procedure.

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