FACTA UNIVERSITATIS (NIŠ)
SER.: ELEC. ENERG. vol. 24, no. 2, August 2011, 147-155

High Order Surface Impedance Boundary Conditions with the A-ϕ Formulation

Dedicated to Professor Slavoljub Aleksić on the occasion of his 60th birthday

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Abstract: Surface impedance boundary conditions (SIBCs) of high order of approximations basd on the Rytov method are introduced and implemented in the A-ϕ finite element formulation. With first order elements, only first and second order approximations for the surface impedance are possible. Third order SIBCs require second order (or higher) finite elements. The order of approximation is not limited but only orders up to three are practical and useful. The method was implemented in an existing FEM code and results are shown to validate its use and accuracy.

Keywords: Surface impedance boundary conditions; finite element analysis; finite element formulations

1 Introduction

SURFACE impedance boundary conditions have been used for the purpose of reduction of the computational space from the very beginning of its introduction. Introduced in 1938 by Schelkunoff [1] and starting with Leontovich in the late 1940's [2], the concept has been in continuous use and has been applied to a vast number of applications, first for analytical solutions and then to numerical computation in all areas, in almost any conceivable formulation and application [3]. However, the approach to surface impedance boundary conditions has been ad-hoc, and almost entirely dependent on the Leontovich condition which is a first

Manuscript received on June 5, 2011.

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order SIBC, that is, the error introduced by its use is directly related to the skin depth. In fact, the error is of the order of $O(\delta^2)$. For this reason, it has found use mostly in high frequency applications and in fact, the original development of the SIBC was for purpose of calculation of propagation effects of radio waves over the surface of the earth. A second order surface impedance boundary condition was developed by Mitzner based on scattering of electromagnetic waves by conducting bodies [4]. However no higher order SIBCs were developed and the use of SIBCs for low frequency applications remained sparse. Nevertheless, a general method for the systematic development of SIBCs of arbitrary order has been introduced by Rytov [5]. Not only does the method due to Rytov include the first and second order conditions (of Leontovich and Mitzner), but it precedes them in time. The method is based on the perturbation approach and is deceptively simple: The electric and magnetic fields were assumed to vary exponentially inside the conductor and were written as power series expansion in terms of the skin depth. Equating equal power of the fields at the surface (in air and in the conductor) provided the appropriate order SIBCs. Because of this, high order SIBCs could be developed by simply retaining additional terms in the expansion [2, 5] and, perhaps more significantly, it provides a systematic method of evaluation of errors, allowing one to match the order of the SIBC with the needs of computation [6]. Newer developments have taken the Rytov process further. In an attempt to extend its applicability and to better define the errors, the expansion was re-defined in terms of the ratio between the skin depth and the characteristic dimension of the conductors (i.e. thickness or radius or, in general, the smallest relevant dimension that will influence the solution. For example, in a thin conductor, of thickness t, the ratio is $p = \delta/t$ and this is required to be small, the smaller, the lower the error introduced by the surface impedance approximation [2, 6].

From an implementation point of view, and especially in implementing the SIBC into existing FEM codes, there are few requirements and these are rather easy to meet. The first and most obvious is that on the impedance surface, the elemental contribution must be modified. However, once a 1st order SIBC has been implemented, higher order SIBCs can be implemented by simply adding terms due to the expansion. This means that the additional work needed is minor. The resulting matrices retain their properties to a large extent (symmetry, conditioning) resulting in a system that is not very different than the system before SIBCs were introduced. However, because conductors are removed from the solution domain, the system is smaller and convergence faster.

This work discusses introduction of 1st and 2nd order SIBCs in an existing finite element code based on the $A-\phi$ formulation. Because the existing code uses only 1st order edge elements, only 1st and 2nd order SIBCs can be implemented. 3rd order SIBCs require 2nd order finite elements.

2 Formulation

One starts with the following two relations in terms of **A** (the magnetic vector potential) and ϕ (the electric scalar potential) [7]:

$$\nabla \times (\frac{1}{\mu}\nabla \times \mathbf{A}) + \sigma(j\omega\,\mathbf{A} + \nabla\phi) = \mathbf{J}_s \tag{1}$$

$$\nabla g\sigma(j\omega\,\mathbf{A} + \nabla\phi) = 0 \tag{2}$$

For linear conductivity one can write instead of (2)

$$\nabla g(j\omega\,\mathbf{A} + \nabla\phi) = 0 \tag{3}$$

It is assumed that any applied current density is in conductors or coils, and that Coulomb's gauge has been applied. Low frequency is assumed. The first equation applies to conducting or nonconducting regions (in the latter case the conductivity is zero) whereas the 2nd equation applies in conducting regions alone, enforcing the zero divergence of induced currents.

The solution domain is assumed to be made of a general domain which encloses magnetic, nonmagnetic, conducting and nonconducting media. Sources can be in the nonconducting space or in the conducting space.

Through use of weighted residuals we get:

$$\int_v [\nabla \times (\frac{1}{\mu}\nabla \times \mathbf{A}) + \sigma(j\omega\,\mathbf{A} + \nabla\phi) - \mathbf{J}_s]\; g\,\mathbf{w}\mathrm{d}v = 0 \tag{4}$$

$$\int_v [\nabla g(j\omega\,\mathbf{A} + \nabla\phi)]\;\; w\mathrm{d}v = 0 \tag{5}$$

where **w** are vector weighting functions and and *w* are scalar weighting functions.

Assuming a finite element mesh divided into edge elements (with properly defined edges, nodes and facets), we approximate the physical quantities as follows:

$$\mathbf{A} = \sum_{e=1}^{N_e} A_e\,\mathbf{w}_e, \qquad \phi = \sum_{n=1}^{N_n} \phi_n\, w_n, \qquad \mathbf{J}_s = \sum_{f=1}^{N_f} J_f \mathbf{w}_f, \tag{6}$$

where *e* stands for the edges of the mesh, *n* for its nodes and f for its facets. $\mathbf{w}_a$ are interpolation functions on edges, w_n are interpolation functions on nodes and $\mathbf{w}_f$ are interpolation functions on facets. N_e is the number of edges (in an element), N_n the number of nodes and N_f the number of facets.

$$\begin{aligned} &\int_v \left[(\frac{1}{\mu}\nabla \times \mathbf{w}_e) g (\nabla \times \mathbf{w}_e)^T + \sigma \mathbf{w}_e g j\omega (\mathbf{w}_e)^T (A_e)^T + \sigma \mathbf{w}_e\, g \nabla w_n \phi_n \mathrm{d}v\right] \mathrm{d}v \\ &\int_\Gamma \mathbf{w}_e\, g[\mathbf{H} \times \hat{\mathbf{n}}]\mathrm{d}\Gamma = \int_v \mathbf{w}_e \sum_{f=1}^{N_f} \mathbf{w}_f J_f \mathrm{d}v \end{aligned} \tag{7}$$

$$\int_v [\sigma \nabla \mathbf{w}_n g j\omega (\mathbf{w}_e)^T (A_e)^T + \sigma \mathbf{w}_e g \nabla w_n \phi_n \mathrm{d}v] \mathrm{d}v \\ - \int_\Gamma [w_n \sigma j\omega \mathbf{w}_e (A_e)^T + w_n \sigma \nabla w_n \phi_n] g\, \hat{\mathbf{n}}\, \mathrm{d}\Gamma = 0 \tag{8}$$

Handling the right hand side contributions due to current densities separately, the elemental matrix looks as follows:

$$\begin{bmatrix} \int_v [\nabla \times \mathbf{w}_e]^T g [\nabla \times \mathbf{w}_e] \mathrm{d}v + j\omega \int_v \sigma [\mathbf{w}_e]^T g [\mathbf{w}_e] \mathrm{d}v & \int_v \sigma [\mathbf{w}_e]^T g [\nabla w_n] \mathrm{d}v \\ \int_v [\nabla w_n]^T g [\mathbf{w}_e] \mathrm{d}v & \frac{1}{j\omega} \int_v \sigma [\nabla w_n]^T g [\nabla w_n] \mathrm{d}v \end{bmatrix} \begin{bmatrix} \mathbf{A} \\ \phi \end{bmatrix} \tag{9}$$

The first term in the matrix is evaluated over the whole volume whereas the other three are only evaluated on the conducting volumes. The surface integrals arising from integration by parts have been set to zero assuming that homogeneous boundary conditions are applied on the outer boundaries. As boundary conditions it is assumed that either the normal **B** or tangential **H** are enforced on the outer boundaries. On conductors, either the tangential component of **E** or the normal component of **J** are enforced. These surface integrals terms will be however re-introduced on the impedance surfaces which can in fact be internal to the mesh or can coincide partially or entirely with the outer boundaries. The surface integrals are only evaluated for the impedance boundary.

The surface integrals are over the whole domain surfaces, including the conductor surface where we will apply the SIBCs. However, since the outer boundaries are already being taken care of, we need not worry about them. Therefore the surface integral above will only apply to the impedance surface.

To introduce the surface impedance, we start with Eq. (9) and write for the surface integral:

$$\int_\Gamma \mathbf{w}_a g [\mathbf{H} \times \hat{\mathbf{n}}] \mathrm{d}\Gamma = \int_\Gamma \frac{1}{\mu} \mathbf{w}_a g [(\nabla \times \mathbf{A}) \times \hat{\mathbf{n}}] \mathrm{d}\Gamma \tag{10}$$

The surface term in Eq. (8) can be set to zero simply from the fact that the normal component of the induced currents must be zero on the impedance surface.

On the impedance surface itself we write the surface current density as:

$$\mathbf{J}_s = \mathbf{H} \times \hat{\mathbf{n}} = \frac{1}{Z_s} [\hat{\mathbf{n}} \times \mathbf{E}] \times \hat{\mathbf{n}} = \frac{1}{Z_s} [\hat{\mathbf{n}} \times (j\omega \mathbf{A} + \nabla \phi)] \times \hat{\mathbf{n}} \tag{11}$$

where Z_s is the surface impedance. The divergence of this current density must be zero:

$$\nabla g \frac{1}{Z_s} [\hat{\mathbf{n}} \times (j\omega \mathbf{A} + \nabla \phi)] \times \hat{\mathbf{n}} = 0 \tag{12}$$

Writing this in the Galerkin form we have on the impedance surface:

$$\int_{\Gamma} \frac{1}{Z_s}\left[\hat{\mathbf{n}} \times (\nabla w_n g j\omega(\mathbf{w}_e)^T (A_e)^T + \nabla w_n g \nabla w_n \phi)\right] \times \hat{\mathbf{n}}\, \mathrm{d}\Gamma = 0 \tag{13}$$

The term $\mathbf{H} \times \hat{\mathbf{n}}$ in Eq. (7) is replaced with the right hand side of Eq. (11) and we have for the surface term in Eq. (7):

$$\int_{\Gamma} \mathbf{w}_a g[\mathbf{H} \times \hat{\mathbf{n}}]\mathrm{d}\Gamma = \int_{\Gamma} \mathbf{w}_e g\left[j\omega(\mathbf{w}_e)^T (A_e)^T + \nabla w_n \phi\right]\mathrm{d}\Gamma \tag{14}$$

Therefore, on the impedance surface the elemental matrix is:

$$\begin{bmatrix} \int_{\Gamma} \frac{1}{Z_s} \mathbf{w}_e g j\omega(\mathbf{w}_e)^T \mathrm{d}\Gamma & \int_{\Gamma} \frac{1}{Z_s} \mathbf{w}_e g \nabla w_n \mathrm{d}\Gamma \\ \int_{\Gamma} \frac{1}{Z_s} \nabla w_n g j\omega(\mathbf{w}_e)^T \mathrm{d}\Gamma & \int_{\Gamma} \frac{1}{Z_s} \nabla w_n g \nabla w_n \mathrm{d}\Gamma \end{bmatrix} \begin{bmatrix} \mathbf{A} \\ \phi \end{bmatrix} \tag{15}$$

This contribution is evaluated on all elements on the impedance surface. For a tetrahedral mesh, the surface elements are triangles.

3 The Surface Impedance

Surface impedance boundary conditions can be developed in a number of ways. The best known surface impedance is due to Leontovich whose surface impedance is no more than the wave impedance in the conductor [2,3] and since that is based on plane wave representation it is suitable for flat or locally flat surfaces. Mitzner [4] developed a second order surface impedance based on scattering by a conducting body suitable for curved surfaces as well while still constraining the propagation of waves in the conductor to the perpendicular direction to the surface. A more general method, one that allows for, in principle, arbitrary order representation of the surface impedance is based on the skin depth and a power series expansion of the field inside the conductor. This method, due to Rytov [5] derives the coefficient of the expansion by equating terms of equal orders in and outside the conductor. Although arbitrary order SIBCs can be derived, it has been shown [3] that the first three order expansion are sufficient for representation of practical configuration whereas higher order become unnecessary because 3rd order SIBCs allow for tangential variations on the surface. The first term of the expansion in the Rytov sequence is the Leontovich SIBC (first order in δ), the second term is the Mitzner SIBC (2nd order in δ), whereas the 3rd order and higher have only been derived in the context of the Rytov approximation. A distinct advantage of this method is the

fact that higher order terms are additive, that is, that addition of a higher order term does not entail any modifications to the previous terms and that the error in using the SIBC can be estimated from the ratio of the skin depth and the characteristic dimension of the geometry [3, 6].

A 3rd order SIBC can be written as the relation between the electric and magnetic field intensities at the surface as follows [3]:

$$(E)^b_{\xi_{3-k}} = (-1)^{3-k} j\omega\mu \begin{bmatrix} (j\omega\sigma\mu)^{-\frac{1}{2}} - (j\omega\sigma\mu)^{-1}\dfrac{d_k - d_{3-k}}{2d_k d_{3-k}} - \\ (j\omega\sigma\mu)^{-\frac{3}{2}}\left(\dfrac{d_k^2 + d_k d_{3-k}^2}{8d_k^2 d_{3-k}^2} - \dfrac{1}{2}\dfrac{\partial^2}{\partial\xi_k^2} + \dfrac{1}{2}\dfrac{\partial^2}{\partial_{3-k}^2} - \dfrac{\partial^2}{\partial_k^2\partial_{3-k}^2}\right) \end{bmatrix} H_k^b,\ k = 1,2 \tag{16}$$

The notation b indicates the boundary, d_1 and d_2 are the radii of curvatures in two orthogonal directions (ξ_1 and ξ_2) on the surface. The negative or positive sign in front of the expression simply indicates the relation between the components of the electric and magnetic fields. The term in the outer brackets, once the derivatives are performed is the surface impedance Z_s required in Eq. (15). Third order SIBCs require 2nd order derivatives and as a matter of implementation in FEM programs require 2nd order elements or higher. However, 1st and 2nd order SIBCs can be implemented with 1st order finite elements. The 2nd order surface impedance is:

$$Z_s = j\omega\mu[(j\omega\sigma\mu)^{-\frac{1}{2}} - (j\omega\sigma\mu)^{-1}\frac{d_k - d_{3-k}}{2d_k d_{3-k}}],\ k = 1,2 \tag{17}$$

The 1st order surface impedance is obtained by removing the 2nd term:

$$Z_s = j\omega\mu(j\omega\sigma\mu)^{-\frac{1}{2}} = \sqrt{\frac{j\omega\mu}{\sigma}} = \frac{1+j}{\sigma\delta} \tag{18}$$

This is immediately recognized as the Leontovich SIBC. In Eq. (17), the surface impedance is curvature dependent. Clearly, if the radii of curvature are equal, the 2nd order SIBC provides identical results to the 1st order SIBC, that is, the surface is viewed as locally flat. The main advantage of this form is when the radii of curvature in the two orthogonal directions are very different such as in cylindrical geometries

4 Implementation and Results

The implementation is straight forward and consists of the following:

1. The elements and nodes belonging to the impedance surface must be identified, usually in a pre-processor.

2. The terms $\mathbf{w}_a$, w_n, and ∇w_n are evaluated in the normal process of assembly although these terms are usually not available for surface elements. If so they need to be defined.
3. The terms of the surface elemental matrix in Eq. (15) are evaluated by the normal process of numerical integration, again, using the triangular elements on the surface.
4. Solution proceeds as normal

As a simple test case, a small coil over a conducting sheet (a classical problem in nondestructive testing) is calculated. The conducting sheet is 3 mm thick but very large in the other dimensions. Its conductivity is 10^7 S/m and its relative permeability 70, representing carbon steel. The coil is small, with a cross-section of 1 mm by 1 mm, with inner radius of 3 mm and placed 1 mm above the sheet (Figure 1). The coil is made of 50 turns carrying a unit current at 100 kHz.

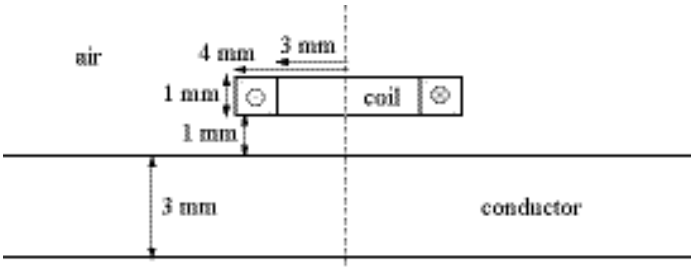

Fig. 1. Geometry of a simple testing configuration.

In NDE, the required output is often in terms of coil impedance or induced voltage in the coil due to variations in the test environment. In the case shown here the flux in the coil was calculated (from which either impedance or induced voltage can be found). The flux with and without the SIBC are compared, with the surface impedance condition imposed on the upper surface of the conductor. The outer boundary coincides with the lower surface of the conductor. The surface impedance was therefore applied to an interior surface, keeping the mesh in both cases identical. Normally, the conductor would be excluded from the mesh entirely - that is one of the reasons for introduction of surface impedance boundary conditions - but in this case it was deemed useful to leave the mesh unchanged. The field distribution in the vicinity of the coil, with and without the SIBC is shown in Figure 2. Although the two figures look similar, there are subtle changes in the field, particularly in the gap between the coil and the conductor. In particular, since the SIBC is internal to the geometry modeled, one can see numerical noise in the volume of the conductor. The fields are very low but, nevertheless, are not null.

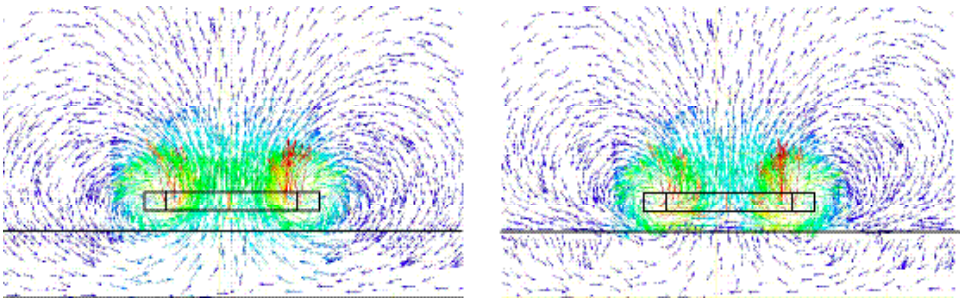

Fig. 2. Field distribution without SIBC (left) and with SIBC (right).

A better indication of the changes in the fields, one that is more relevant is the change in flux in the coil due to the imposition of the SIBC. These changes are shown in Table 1, for various conductivities. As expected, the error should increase with the decrease in conductivity since as the frequency decreases the skin depth increases and hence the ratio between skin depth and the characteristic dimension of the geometry decreases. In this case the characteristic dimension is the thickness of the conducting sheet (3 mm). The values for flux in Table 1 are magnitudes. Table 1 also indicates the error levels one can expect due to the approximation with SIBCs. This error will depend on a number of parameters including the distance between the coil and the surface on which the SIBC is imposed, skin depth and mesh density.

Table 1. Flux and error in calculation of flux in the coils as a function of conductivity of the plate.

Conductivity	Flux without SIBC	Flux with SIBC	% change (error)
10^7 S/m	9.33948	9.22142	1.28
10^6 S/m	9.49206	9.22142	2.93
10^5 S/m	9.57937	9.22142	3.88
10^4 S/m	9.58843	9.221419	3.99

The results shown here do not actually benefit from the second order SIBC since the surface is flat and the second term in Eq. (18) is zero. However for curved surfaces the term should make a significant difference as was shown elsewhere [1, 6]. It should also be noted in passing that the convergence of the solution is approximately twice as fast even though the number of unknowns remains exactly the same. This of course will vary from one application to the other but considering the fact that SIBCs allow reduction of the number of unknowns (in some cases by a significant ratio), one of the advantages of using SIBCs is this overall reduction in computational effort and resources. Finally, although the present formulations is in the frequency domain, equivalent time-domain formulations can also be implemented.

5 Conclusions

The results shown here point to the validity and value of implementing surface impedance boundary conditions in FEM codes. Their introduction requires little extra effort and incurs small errors for a significant benefit. The introduction of second or third order SIBC has been shown to be additive with essentially no extra computational effort. The effect of high curvatures remains to be evaluated in future work as are SIBCs for other formulations such as the T - Ω method.

References

[1] S. A. Shelkunoff, "The impedance concept and its application to problems of reflection, radiation, shielding and power absorption," *Bell Syst. Tech. J.*, vol. 17, pp. 17–48, 1938.

[2] M. A. Leontovich, "On the approximate boundary conditions for the electromagnetic field on the surface of well conducting bodies," in *Investigations of Radio Waves*, B. A. Vvedensky, Ed. Moscow: Acad. of Sciences of USSR, 1948.

[3] S. Yuferev and N. Ida, *Surface Impedance Boundary Conditions*. Boca Raton: CRC Press, June 2010, iSBN 978-1-4200-4489-8.

[4] K. M. Mitzner, "An integral equation approach to scattering from a body of finite conductivity," *Radio Science*, vol. 2, no. 12, pp. 1459–1470, 1967.

[5] S. M. Rytov, "Calcul du skin-effet par la methode des perturbations," *Journal of Physics*, vol. II, no. 3, 1940.

[6] S. Yuferev and N. Ida, "Selection of the surface impedance boundary conditions for a given problem," *IEEE Transactions on Magnetics*, vol. 35, no. 3, pp. 1486–1489, May 1999.

[7] Y. L. Menach, "Contribution a la modelisation numerique tridimensionnelle des systemes electrotechniques," Ph.D. dissertation, Universite de Lille 1, 1999.