Implementation of High-Order On-Surface Radiation Boundary Conditions

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Electromagnetic scattering problems that involve far-field radiation patterns and the calculation of total currents induced in a perfect conductor can be solved using local radiation boundary conditions. These local conditions are often imposed on a domain enclosing the scatterer, and the typical finite-element methods are incorporated so that the well-posedness of the modified problems encompassing the local radiation boundary conditions is preserved. Much effort in recent years has been devoted to attempts to construct higher order far-field conditions, so that the solution accuracy can be improved. In this article, we avoid extensive computations and bring the radiation boundary on the scatter's surface itself. This procedure is known as the on-surface radiation boundary condition (OSRBC). The limitations in the past have been the implementable order of the OSRBC. Nevertheless, the key feature of the OSRBC to calculate the relevant quantities for engineers is the normal derivative of the solution on the OSRBC. This article introduces a new method for calculating the normal derivative of the electric field on the surface of a scatterer of the known shape. The method is based on a formulation of the boundary conditions through a recursive sequence of differential operators. The numerical implementation of this formulation allows one to extract a relation which is then used to solve for the quantity of interest, such as radar cross section.

Index Terms—**Boundary conditions, scattering, on-surface radiation boundary condition (OSRBC).**

I. Introduction

ELECTROMAGNETIC scattering problems are most often too difficult to solve analytically and usually call for numerical and/or asymptotic methods. Computational electromagnetics, particularly applied to radar cross sections, often involves complex and time-consuming numerical algorithms (typically finite-element computation). These algorithms have an accuracy that relies on the choice of certain parameters (e.g., the elements) and on an accurate prescription of far-field conditions. For better computational performance, it is useful to truncate the computational domain with an artificial boundary. Bayliss and Turkel [1] constructed a sequence of radiating boundary conditions and proved the convergence of the solution toward the infinite domain solution as the boundary is moved far from the scatterer. Kriegsmann *et al.* [7] have shown that it is possible to bring the radiation boundary on the scatterer boundary Γ and still calculate the radar cross section. The advantage of this approach is that the domain of integration is reduced to the contour of the scatterer. Specifically, this principle involves calculation of the normal derivative on Γ. The principle is referred to as on-surface radiation boundary condition (OSRBC). Unfortunately, the closer one brings the radiation boundary to the scatterer, the more accurate the prescription of the radiation condition must be. Kriegsmann *et al.* [7] used either first- or second-order radiation conditions. In addition, their work was limited to frequency domain problems because the OSRBCs were based on high-frequency operators (either global as in [1] or local in [7]). To address problems at low-frequencies, which arise in some applications such as non-destructive testing, or to study transients of physical systems, one needs to develop other tools. In this article, we present arbitrary order OSRBCs in the time domain, from which one can obtain the frequency domain results as well.

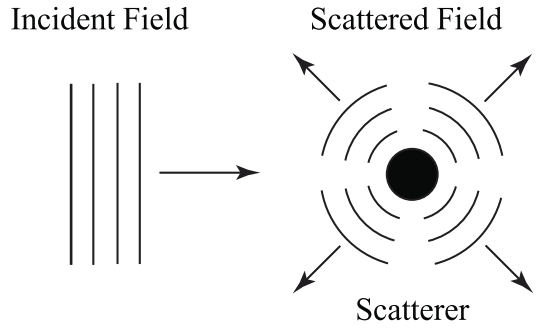

Fig. 1. Geometry of the problem.

The original formulation of on-surface radiation conditions that we use appeared in [5]. This formulation relies on the theoretical formulation of the scattered field solution presented in [2]–[4]. Hariharan *et al.* [6] presented an implementation of this formulation. We propose to continue this article by extending it with an implementation of higher order OSRBCs. Implementation of these time-domain conditions to a higher order should yield improved accuracy.

The problem considered here is an exterior Dirichlet problem for a wave external to a region Ω (the scatterer) bounded by a boundary of arbitrary shape Γ. The problem is illustrated in Fig. 1. The situation is governed by the wave equation

$$\nabla^2 u - \mu\varepsilon \frac{\partial^2 u}{\partial t^2} = 0 \tag{1}$$

where u is the field. Boundary conditions needed to solve the problem are demonstrated in [5] and take the form of a recursively defined sequence of equations. They are expressed

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in polar coordinates as follows:

$$\frac{1}{c}\frac{\partial u}{\partial t}+\frac{\partial u}{\partial r}+\frac{1}{2r}u=\omega_1 \tag{2}$$

$$\frac{1}{c}\frac{\partial \omega_j}{\partial t}+\frac{j}{r}\omega_j=\frac{(j-1/2)^2}{4r^2}\omega_{j-1}+\frac{1}{4r^2}\frac{\partial^2\omega_{j-1}}{\partial\theta^2}+\omega_{j+1}\quad \forall\, j\in N^* \tag{3}$$

where the functions ω_j are auxiliary functions defined recursively called *remainders*, $\omega_0=2u$ and $c^2=\mu\varepsilon$. For $\omega_1=0$, the condition is known as the Baylis and Turkel condition. Higher order implementations are introduced here.

II. Proposed Method

It has been shown in [5] that $\omega_j=O_{r\to\Gamma}(r^{1/2-2j})$. For each integer j, the function ω_j is called the *remainder of order* j. As the value of a remainder on the boundary Γ decreases when its order increases (for values of r on Γ greater than 1, i.e., for Γ enclosing the unit circle), it is natural to set some order p that one can choose according to the desired accuracy, beyond which the remainder can be neglected. From this approximation, (3) can be solved backward in j to find ω_1 in terms of u, and then (2) provides the radial derivative $\partial u/\partial r$. The radial derivative can then be used to calculate $\partial u/\partial n$

$$\nabla\times\mathbf{n}=\begin{pmatrix}\partial/\partial r\\(1/r)\partial/\partial\theta\end{pmatrix}\times\begin{pmatrix}n_1\\n_2\end{pmatrix}=n_1\frac{\partial}{\partial r}+n_2\frac{1}{r}\frac{\partial}{\partial\theta}.$$

The process of finding $\partial u/\partial r$ using the conditions (2) and (3) is the contribution of this article.

III. Implementation of the Method

For each and every j, we have: $\omega_j=O_{r\to\Gamma}\left(r^{1/2-2j}\right)$, and we may assume that beyond some given index (fixed arbitrarily depending on the desired accuracy), the remainder becomes so small that it can be assumed to be 0. Denoting ω_p the last non-zero remainder, that is, all remainders ω_j with $j>p$ are neglected and assumed to be zero. Next, we discretize (3) and write this in the form

$$\frac{-1}{2cdt}\omega^j_{n,k-1}+\frac{j}{r}\omega^j_{n,k}+\frac{1}{2cdt}\omega^j_{n,k+1}-\omega^{j+1}_{n,k}=\frac{1}{4r^2d\theta^2}\omega^{j-1}_{n-1,k}+\frac{(j-1/2)^2d\theta^2-2}{4r^2d\theta^2}\omega^{j-1}_{n,k}+\frac{1}{4r^2d\theta^2}\omega^{j-1}_{n+1,k} \tag{4}$$

where

$$\begin{cases}d\theta=\frac{2\pi}{N-1}, & \text{where } N \text{ is an integer}\\ dt=\frac{t_{\text{final}}}{K-1}, & \text{where } K \text{ is an integer}\\ \omega^j_{n,k}=\omega_j(r,nd\theta,kdt), & \text{where } \begin{vmatrix}k\in\{0,1,2,\ldots,K-1\}\\ n\in\{0,2,\ldots,N-1\}.\end{vmatrix}\end{cases}$$

To find a simple relation between the remainders ω_j in (4), we introduce the notation: $\omega^j_{n,k}=\omega_j(r,nd\theta,kdt)$, and define

$$Z^j=[\omega^j_{1,1},\omega^j_{1,2},\ldots,\omega^j_{1,K},\omega^j_{2,1},\ldots,\omega^j_{2,K},\ldots,\omega^j_{N,K}]^{\mathrm{T}}. \tag{5}$$

As a consequence, (4) becomes

$$M_jZ^{j-1}=T_jZ^j-Z^{j+1}$$

where M_j and T_j are $NK\times NK$ matrices given by

$$M_j=\begin{pmatrix}A_j & B & 0 & \cdots & B\\ B & A_j & B & \ddots & \vdots\\ 0 & B & \ddots & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & B\\ B & \cdots & 0 & B & A_j\end{pmatrix} \tag{6}$$

$$T_j=\begin{pmatrix}C_j & 0 & \cdots & \cdots & 0\\ 0 & C_j & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & 0\\ 0 & \cdots & \cdots & 0 & C_j\end{pmatrix} \tag{7}$$

where A_j, B, and C_j are three $K\times K$ matrices defined by

$$A_j=\frac{-2+(j-1/2)^2d\theta^2}{4r^2d\theta^2}I_K \tag{8}$$

$$B=\frac{1}{4r^2d\theta^2}I_K \tag{9}$$

$$C_j=\begin{pmatrix}\frac{j}{r} & \frac{1}{2cdt} & 0 & \cdots & 0\\ \frac{-1}{2cdt} & \frac{j}{r} & \frac{1}{2cdt} & \ddots & \vdots\\ 0 & \frac{-1}{2cdt} & \ddots & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & \frac{1}{2cdt}\\ 0 & \cdots & 0 & \frac{-1}{2cdt} & \frac{j}{r}\end{pmatrix}. \tag{10}$$

We have chosen p such that $Z_j=0$ for $j>p$ and, in particular,

$$\begin{cases}Z^{j-1}=M_j^{-1}T_jZ^j-M_j^{-1}Z^{j+1}\\ Z^{p-1}=M_p^{-1}T_pZ^p.\end{cases}$$

It is useful to write

$$Z^p=P_pZ^p\quad\text{and}\quad Z^{p-1}=P_{p-1}Z^p$$

where matrices P_p and P_{p-1} are given by

$$P_p=I_{NK}\quad\text{and}\quad P_{p-1}=M_p^{-1}T_p.$$

It can be proven that a similar relation holds for all orders. For example, one needs to show that for $m\le p$, there exist P_m's such that $Z^m=P_mZ^p$. To do so, we first assume this to be true for some integer $m<p$ and all orders from m to p (it is true for $m=p-1$ as shown previously). Then

$$Z^{m-1}=M_m^{-1}T_mZ^m-M_m^{-1}Z^{m+1}=M_m^{-1}T_mP_mZ^p-M_m^{-1}P_{m+1}Z^p=\underbrace{\left(M_m^{-1}T_mP_m-M_m^{-1}P_{m+1}\right)}_{P_{m-1}}Z^p.$$

By induction, for any integer $m\in\{1,2,\ldots,p\}$

$$Z^m=P_mZ^p. \tag{11}$$

In particular,

$$\begin{cases}Z^1=P_1Z^p\\ Z^0=P_0Z^p.\end{cases}$$

A recursive definition of the matrices P_m's has been established, allowing their calculation

$$\begin{cases} P_p = I_{NK} \\ P_{p-1} = M_p^{-1} T_p \\ P_{m-1} = M_m^{-1}(T_m P_m - P_{m-1}), \quad m \in \{2, \ldots, p-1\}. \end{cases}$$

P_0 and P_1 are of interest here since these are the quantities needed to calculate $\partial u/\partial r|_{r\in\Gamma}$.

Using the notation

$$u_{n,k} = u(r \in \Gamma, nd\theta, kdt), \quad v_{n,k} = \partial u/\partial r(r \in \Gamma, nd\theta, kdt)$$

two vectors U and V are defined as

$$U = [u_{1,1}, u_{1,2}, \ldots, u_{1,K}, u_{2,1}, \ldots, u_{2,K}, \ldots, u_{N,K}]^{\mathrm{T}}$$
$$V = [v_{1,1}, v_{1,2}, \ldots, v_{1,K}, v_{2,1}, \ldots, v_{2,K}, \ldots, v_{N,K}]^{\mathrm{T}}.$$

The goal is to determine V. The discretization of (2) becomes

$$\frac{1}{c}\frac{u_{n,k+1} - u_{n,k-1}}{2dt} + v_{n,k} + \frac{1}{2r}u_{n,k} = \omega^1_{n,k}. \tag{12}$$

Using a matrix formulation, this is written as

$$\frac{1}{2cdt}SU + V + \frac{1}{2r}U = P_1 Z^p \tag{13}$$

where S is an $(N \times K) \times (N \times K)$ matrix given by

$$S = \begin{pmatrix} S_k & 0 & 0 & \cdots & 0 \\ 0 & S_k & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & S_k \end{pmatrix}$$

with

$$S_k = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \ddots & \vdots \\ \vdots & -1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & -1 & 0 \end{pmatrix} (aK \times K \text{matrix}).$$

The result (11) is inverted and substituted into the following:

$$Z^0 = 2U = P_0 Z^p \Rightarrow Z^p = 2P_0^{-1}U.$$

This yields an expression of the radial derivative that can be used for computation of vector V

$$V = \left[2P_1 P_0^{-1} - \frac{1}{2r}I_{NK} - \frac{1}{2cdt}S\right]U.$$

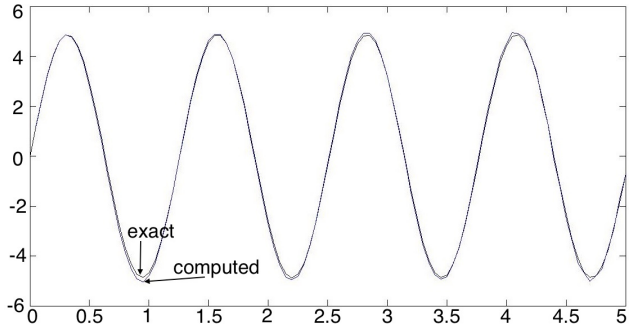

Fig. 2. $du/dn(R, 0, t)$ with respect to t (for $t_{\text{final}} = 5$).

IV. Results

The method has been implemented for a circular scatterer and for the incident field

$$u_{\text{inc}} = \text{Re}\{e^{-i\omega t}e^{\beta r \cos\theta}\}$$

where β has been taken to be 5. The exact solution used for comparison and verification was calculated using potential theory [8]–[10] but its details are not given here because it is lengthy. The solution generated by the method introduced in this article and the exact solution are shown in Fig. 2 for two given time intervals.

The results are practically indistinguishable from the exact solution, thereby validating the method. Although it is not possible to see that in the figure, the first and last point in the plot diverges from the exact solution. This is because a leap-frog scheme was used for the time derivative to improve accuracy. The exact solution used for comparison is limited to simple geometries (in the case shown in Fig. 2 for a cylindrical boundary). More complex boundaries can also be evaluated and compared with experimental data such as may be obtained from radar cross-section measurements. The analytical solution to more complex geometries is not usually possible (with the exception, perhaps, of elliptical cross-sectional scatterers), and therefore, comparison can only be done with experimental data. However, for the validation of the method, which is the purpose of the result given here, an analytic solution is preferable and the results have shown, while limited in scope, to verify the method.

V. Conclusion

We have demonstrated that the calculation of the radial derivative is possible without the use of sophisticated and expensive numerical algorithms. A simple finite difference scheme allowed the calculation of the radial derivative to any desired accuracy. This has been achieved by exploiting an OSRBC formulation available in the literature. The method has been validated on the example of a circular scatterer illuminated by a plane wave. Scatterers of different shapes may be used as long as a polar description of their boundaries is available. Different incident fields may also be used. In addition, the use of the Laplace transform (in time) reduces drastically the size of the systems to manipulate and therefore improves the computational time.

However, the cost is an additional numerical error due to the Laplace transformation and inversion. Thus, one must decide if higher accuracy is more important or if shorter computational time is needed.

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