High Order Differential Form-Based Elements for the Computation of Electromagnetic Field

Z. Ren, *Senior Member, IEEE*, and N. Ida, *Senior Member, IEEE*

***Abstract*—The Whitney elements, discrete spaces based on differential forms, have proven their efficiency in electromagnetic field computation. However, they are built only in first order. This paper gives a general description of high order p-form (nodal, edge, facet and volume) elements. Their function spaces and the assignment of degrees of freedom on the simplexes are analyzed. General expressions of the basis functions are given. A comparison of several 2nd order elements is carried out. A procedure for the generation of hierarchical basis of p-form element is provided.**

***Index Terms*—Differential forms, edge elements, finite element modeling, high order elements.**

I. Introduction

THE CALCULUS of differential forms is a useful tool to describe physical phenomena such as electromagnetics [1], [2]. It constitutes a natural framework for the description of electromagnetic theory and has numerous advantages compared to the conventional vector algebra. The appearance of Whitney elements (first order differential form-based elements) [3] was a considerable advance in the finite element computation of electromagnetic fields. Whitney elements consider the differential forms as degrees of freedom. Their advantages are principally their capacity of allowing natural discretization of the systems with appropriate continuity of scalar and vector variables. However, Whitney elements are built only in first order. To increase the accuracy of interpolation, high order differential form based elements must be introduced. The theory of high order edge (curl-conformal) and facet (div-conformal) elements was advanced in the early 80's in [4]. Unfortunately, in this reference, no specific vector basis function was reported.

Further investigation has been carried out in recent years by different researchers [5]–[10]. Most of those works focus on the high order edge element. Few studies on the link between nodal, edge, facet and volume elements (differential form based elements of different degrees) were carried out. In this paper, we give a general description of high order differential form based elements starting from De Rham's complex. Analysis of their function spaces and of the assignment of degrees of freedom (D.O.F.) on each simplex will be reported. General expressions of basis functions fulfilling the conformity requirement will be given. Several 2nd order edge elements will be compared. It should be noted that it is not our objective to give concrete element bases, but through this analysis, to try to provide a general procedure to generate those bases.

We adopt in this paper the term "differential form based elements" or "p-form elements" rather than the classical names "nodal, edge, facet or volume elements" which are originally related to where the D.O.F. are assigned, but no longer appropriate for high order cases. Because, for example, in the case of high order 1-form elements, the D.O.F. are assigned not only to edges but can also be to facets and volume. Other suitable terms might be "tangential vector element" instead of "edge element" and "normal vector element" instead of "facet element."

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Z. Ren is with Ansoft Corporation, 4 Station Square, Pittsburgh, PA 15220 USA. He is on leave from Laboratoire de Génie Electrique de Paris, C.N.R.S., France (e-mail: zren@ansoft.com).

N. Ida is with the Department of Electrical Engineering, the University of Akron, Akron, OH 44325 USA (e-mail: ida@uakron.edu).

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II. Differential Forms and De Rham's Complex

Differential forms are expressions on which integration operates [1]. A differential form of degree p, or a p-form, is an expression where the integral is performed over a manifold of dimension p in a space of dimension n, i.e. the integrand of a p-fold integral in an n-dimensional space. In electromagnetism, according to the dimension of the manifold on which the variable is integrated, a scalar potential is a 0-form; the circulation of a vector potential or a field intensity (electric or magnetic) along a small segment is a 1-form, a flux (or current) across a small area is a 2-form and charges contained in a small volume is a 3-form.

Differential forms operate in exterior algebra. Exterior (wedge) product of a p-form ω and a q-form υ produces a $(p+q)$-form $(p+q<n)$ with the skew symmetry property: $\omega \wedge \upsilon = (-1)^{pq} \upsilon \wedge \omega$. Two other operators permit transformation of a differential form of one degree to the other. One is the exterior derivation "d." Application of this operator to a differential form leads to a form of higher degree. In three dimensions, it replaces the familiar "grad," "curl" and "div" operators of vector algebra. The other operator is the star (Hodge) operator "$*$." It transforms a p-form to an $(n\text{–}p)$-form, where n denotes the dimension of space.

Let $D^p(M)$ be the set of p-forms defined on an n-dimensional manifold M, we have $dD^p(M) \subset D^{p+1}(M)$. This property can be represented by a sequence called De Rham's complex [2]. The case of $n=3$ is shown in Fig. 1.

A form ω is said to be closed if $d\omega = 0$. A form ω is said to be exact if there exists a form υ (of one degree lower) such that $\omega = d\upsilon$. Since $d(d\upsilon) \equiv 0$, every exact form is closed. Can we also say "every closed form is exact?" According to the Poincare lemma, the answer is positive for a manifold not too complex (topologically trivial domains). But in general, the answer is negative. Let $Z^p(M)$ be the set of closed

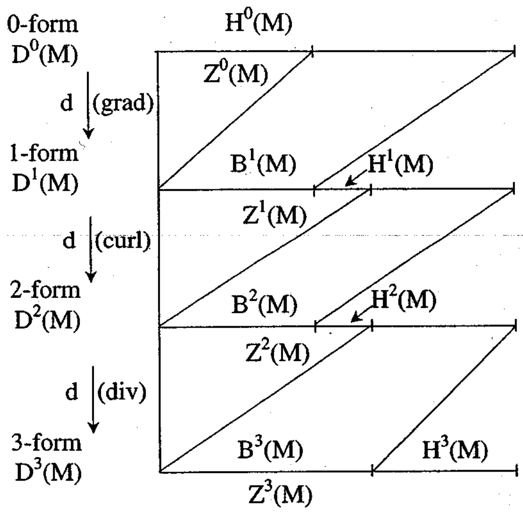

Fig. 1. De Rham's complex in the case of 3 dimensions.

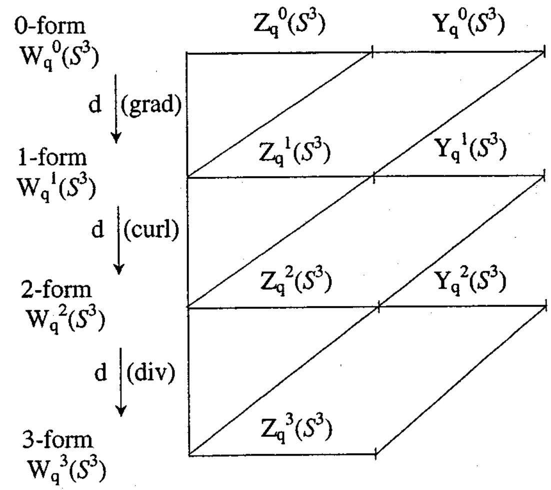

Fig. 2. De Rham's complex showing the relation between p-form elements.

p-forms, $B^p(M)$ be the set of exact p-forms. We have in general $B^p(M) \subset Z^p(M)$. The complement of $B^p(M)$ in $Z^p(M)$, $H^p(M) = Z^p(M) \backslash B^p(M)$ is called De Rham's pth cohomology group. Dimension of H^p depends on the topology of the manifold [11]. H^p (for $p > 0$) vanishes on a trivial manifold.

In order to model correctly and naturally the spaces of differential forms of different degrees, suitable elements must be adopted. These elements can be derived with the help of the calculus of differential forms, notably the De Rham's complex.

III. Function Spaces of Differential Form-Based Elements

Let S^3 be a 3-simplex (tetrahedron), $W_q^p(S^3)$ the function space of q order p-form elements constructed on S^3. W_q^p can be decomposed into a null space of differential operator $Z_q^p(S^3)$ (set of closed forms) and a range space of differential operator $Y_q^p(S^3)$: $W_q^p = Z_q^p \oplus Y_q^p$. The tetrahedron is topologically trivial. The closed forms are exact forms. This property is shown with the help of De Rham's complex in Fig. 2. It can be seen that the differential operator is an isomorphism of Z_q^p onto Y_q^{p-1} and $\dim(Y_q^{p-1}) = \dim(Z_q^p)$.

The element W_q^p must fulfill the following requirements: model correctly the null space Z_q^p of the differential operator and be complete to $q-1$ order in the range space Y_q^p under the differential operation. In order to define the number of D.O.F. of q-order p-form element W_q^p, we analyze the dimension of subspaces Z_q^p and Y_q^p. We denote by $P_q(S^3)$ the linear space of polynomials on S^3 with degree up to q.

The 0-form element space W_q^0 is spanned by q-order polynomials: $W_q^0 = P_q$. Its dimension is $(q+1)(q+2)(q+3)/6$. The space Z_q^0 is a constant field with the dimension 1. The space Y_q^0 is spanned by q order polynomials with nonzero gradients. Its dimension is obviously $(q+1)(q+2)(q+3)/6-1$. The dimension of Y_q^0 can also be determined in the following way: Y_q^0 is complete to $q-1$ order under the grad operation. The number of D.O.F. of a $q-1$ order polynomial vector is $q(q+1)(q+2)/2$. The curl free condition gives $q(q-1)(2q+5)/6(\dim(P_{q-2})^3 - \dim(P_{q-3}))$ constraint relations. Hence, the dimension of the space Y_q^0 is $(q+1)(q+2)(q+3)/6-1$.

The space Z_q^1 is a gradient field. Its dimension is $(q^3+6q^2+11q)/6$, equal to that of Y_q^0. The space Y_q^1 is a curl field of q order polynomial. It must be complete to $q-1$ order under the curl operation. The number of D.O.F. of a $q-1$ order polynomial vector is $q(q+1)(q+2)/2$. There are $q(q-1)(q+1)/6(\dim(P_{q-2}))$ relations to ensure the divergence free condition. Hence, the dimension of the space Y_q^1 is $q(q+1)(2q+7)/6$. The total number of degrees of freedom of the 1-form element space W_q^1 is $q(q+2)(q+3)/2$.

The space Z_q^2 is a curl field which has the same dimension as Y_q^1: $(2q^3+9q^2+7q)/6$. The space Y_q^2 is a divergence field of q order polynomial. It must be complete to $q-1$ order under the div operation. The number of D.O.F. of a $q-1$ order polynomial scalar field is $q(q+1)(q+2)/6$. This defines the dimension of the space Y_q^2. The total number of degrees of freedom of the 2-form element space W_q^2 is $q(q+1)(q+3)/2$.

The 3-form element space W_q^3 has the same dimension as Y_q^2 and is spanned by polynomials of order up to $q-1$. The number of D.O.F. of W_q^3 is $\dim(P_{q-1}) = q(q+1)(q+2)/6$.

The dimensions of spaces $Z_q^p(S^3)$, $Y_q^p(S^3)$ and $W_q^p(S^3)$ of p-form elements are summarized in Table I.

Let us introduce the following spaces defined over S^3:

$$\begin{aligned}
&\tilde{P}_q\text{: linear space of homogenous polynomials of degree } q,\\
&\tilde{G}_q = \left\{ \boldsymbol{v} \in (\tilde{P}_q)^3 | \boldsymbol{v} = \mathrm{grad}\phi,\ \phi \in \tilde{P}_{q+1} \right\},\\
&\tilde{S}_q = \left\{ \boldsymbol{v} \in (\tilde{P}_q)^3 | \boldsymbol{r} \cdot \boldsymbol{v} = 0 \right\}.
\end{aligned}$$

These spaces are related by the Helmholtz decomposition: $(\tilde{P}_q)^3 = \tilde{G}_q \oplus \tilde{S}_q$. Their dimensions are, respectively,

$$\begin{aligned}
\dim(\tilde{P}_q) &= (q+1)(q+2)/2,\\
\dim(\tilde{G}_q) &= \dim(\tilde{P}_{q+1}) = (q+2)(q+3)/2,
\end{aligned}$$

and

$$\dim(\tilde{S}_q) = q(q+2).$$

TABLE I
DIMENSION OF FUNCTION SPACES OF Q-ORDER P-FORM ELEMENTS

Spaces / elements	$Z_q^p(S^3)$ (null space of d)	$Y_q^p(S^3)$ (range space of d)	$W_q^p(S^3)$ (function space)
0-form	1	$\frac{(q+1)(q+2)(q+3)-6}{6}$	$\frac{(q+1)(q+2)(q+3)}{6}$
1-form	$\frac{(q+1)(q+2)(q+3)-6}{6}$	$\frac{q(q+1)(2q+7)}{6}$	$\frac{q(q+2)(q+3)}{2}$
2-form	$\frac{q(q+1)(2q+7)}{6}$	$\frac{q(q+1)(q+2)}{6}$	$\frac{q(q+1)(q+3)}{2}$
3-form	$\frac{q(q+1)(q+2)}{6}$	-	$\frac{q(q+1)(q+2)}{6}$

Let us define further:

$$\tilde{C}_q = \left\{ v \in (\tilde{P}_q)^3 | \boldsymbol{v} = \mathrm{curl}\boldsymbol{u},\, \boldsymbol{u} \in \tilde{S}_{q+1} \right\}$$

According to De Rham's complex, the curl operator is an isomorphism of $\tilde{S}_{q+1}$ onto $\tilde{C}_q$, and $\dim(\tilde{C}_q) = \dim(\tilde{S}_{q+1}) = (q+1)(q+3)$.

Following the above definitions, the null spaces of p-form elements are decomposed as $(q \geq 2)$:

$$\begin{aligned} &Z_1^0 = \tilde{P}_0 = 1, && Z_q^0 = Z_{q-1}^0, \\ &Z_1^1 = \tilde{G}_0 = 3, && Z_q^1 = Z_{q-1}^1 \oplus \tilde{G}_{q-1} = G_{q-1}, \\ &Z_1^2 = \tilde{C}_0 = 3, && Z_q^2 = Z_{q-1}^2 \oplus \tilde{C}_{q-1} = C_{q-1}, \\ &Z_1^3 = \tilde{P}_0 = 1, && Z_q^3 = Z_{q-1}^3 \oplus \tilde{P}_{q-1} = P_{q-1}. \end{aligned}$$

Where G_{q-1} is the subspace of $(P_{q-1})^3$ which contains gradient vectors and C_{q-1} is the complement of G_{q-1} in $(P_{q-1})^3$ containing nonzero curl vectors. Consequently, the function spaces of p-form elements have the decomposition $(q \geq 2)$:

$$\begin{aligned} &W_1^0 = P_1, && W_q^0 = W_{q-1}^0 \oplus \tilde{P}_q = P_q, \\ &W_1^1 = \tilde{G}_0 \oplus \tilde{S}_1, && W_q^1 = W_{q-1}^1 \oplus \tilde{G}_{q-1} \oplus \tilde{S}_q, \\ &W_1^2 = \tilde{C}_0 \oplus \tilde{P}_0, && W_q^2 = W_{q-1}^2 \oplus \tilde{C}_{q-1} \oplus \tilde{P}_{q-1}, \\ &W_1^3 = \tilde{P}_0, && W_q^3 = W_{q-1}^3 \oplus \tilde{P}_{q-1} = P_{q-1}. \end{aligned}$$

IV. COMPARISON WITH COMPLETE Q-ORDER ELEMENTS

The previously given function spaces W_q^p of q-order differential form-based elements are complete to $q-1$ order under the differential operation but incomplete themselves to q-order (except for 0-form elements), because their dimensions are smaller than that of complete q-order vector or scalar basis. The reason is that the null spaces $Z_q^p (p = 1, 2, 3)$ are only complete to $q-1$ order.

To get complete q-order elements, it is enough to complete the null space Z_q^p to q-order by adding, respectively, $\tilde{G}_q$ to Z_q^1, $\tilde{C}_q$ to Z_q^2 and $\tilde{P}_q$ to Z_q^3. Let us denote by N_q^p the function spaces of complete q-order p-form elements, we have the following relationships between N_q^p and W_q^p:

$$N_q^1 = W_q^1 \oplus \tilde{G}_q, \quad N_q^2 = W_q^2 \oplus \tilde{C}_q, \quad N_q^3 = W_q^3 \oplus \tilde{P}_q$$

It should be noted that N_q^1 and N_q^2 are the family of vector elements given in [12].

Since $\tilde{G}_q$ is a gradient space, adding $\tilde{G}_q$ to the function space of 1-form elements results only in additional irrotational functions. This will not contribute to the modeling of range space of the curl operator and there will be no influence on the accuracy of rotational fields.

Similarly, $\tilde{C}_q$ is a curl space. Adding $\tilde{C}_q$ to the 2-form elements space results only in additional solenoidal functions and has no influence on the accuracy of divergence fields.

These observations show that the 1-form (or 2-form) element of complete order allow a better approximation of the vector field but don't affect the accuracy of the curl (or div) field. In the case where we are mostly interested in the curl field, for example in the case of magnetostatic field using the vector potential formulation, the use of incomplete q-order element is more economical and hence preferable.

We can also note that, if we take a gradient space greater than G_{q-1} but smaller than G_q, we get elements with the number of D.O.F. smaller than $\dim(N_q^p)$ but greater than $\dim(W_q^p)$. The element given in [10] is such an example.

In this paper, we are interested only in elements of incomplete q-order W_q^p. The analysis is similar for the case of complete q-order elements.

V. ASSIGNMENT OF DEGREES OF FREEDOM

The number of D.O.F. of p-form elements has been given. In this section, we will answer the question how to determine the number of D.O.F. to be assigned to r-simplex S^r ($r = 0$: vertex, $r = 1$: edge, $r = 2$: facet and $r = 3$: volume).

A. 0-Form Element (Nodal Element)

When the order of 0-form element is upgraded from $q-1$ to $q\,(q \geq 2)$, the number of D.O.F. to be added to an edge is 1, to a facet is $(q-2)$, to a volume is $(q-2)(q-3)/2$, and the total is $\dim(\tilde{P}_q)$. Let us denote by P_q^r: the spaces of polynomials of degree q on an r-simplex S^r $(r = 0, 1, 2, 3)$ which vanish on the boundary, ∂S^r $(r = 1, 2, 3)$. We have:

$$W_q^0 = P_q = 4 \times P_q^0 \oplus 6 \times P_q^1 \oplus 4 \times P_q^2 \oplus P_q^3$$

The D.O.F. of q-order nodal element $W_q^0(S^3)$ to be assigned to vertices, edges, facets and volume are, respectively, $4\times\dim(P_q^0)=4\times 1, 6\times\dim(P_q^1)=6\times(q-1), 4\times\dim(P_q^2)=4\times(q-2)(q-1)/2, \dim(P_q^3)=(q-3)(q-2)(q-1)/6$, so that the total is $(q+1)(q+2)(q+3)/6$.

B. 1-Form Element (Edge Element)

In order to determine the number of D.O.F. of 1-form element on each simplex, we must know the dimension of gradient and curl spaces on the simplexes. Let us denote by

- $P_q(S^r)$: the linear space of polynomials defined on S^r with degree up to q.
- U_q^r: the 1-form space of polynomials with degree up to q on a p-simplex $S^r(r=1,2,3)$ which vanish on the boundary $\partial S^r(r=2,3)$ (1-vectors with zero tangential components on ∂S^r).
- G_q^r: the subspace of U_q^r containing exact (closed) 1-forms (vectors which are gradients of a polynomial of degree $q+1$).
- S_q^r: the complement of G_q^r in U_q^r (vectors having nonzero curl).

We have $P_q(S^1)=U_q^1$, $(P_q(S^2))^2=3\times U_q^1\oplus U_q^2$ and $(P_q(S^3))^3=6\times U_q^1\oplus 4\times U_q^2\oplus U_q^3$ with $\dim(P_q(S^1))=q+1$, $\dim(P_q(S^2))^2=(q+1)(q+2)$ and $\dim(P_q(S^3))^3=(q+1)(q+2)(q+3)/2$. Hence, the dimension of U_q^p can be derived: $\dim(U_q^1)=q+1$, $\dim(U_q^2)=(q-1)(q+1)$ and $\dim(U_q^3)=(q-2)(q-1)(q+1)/2$.

Otherwise, according to De Rham's complex, the grad operator is an isomorphism of P_{q+1}^r onto G_q^r, we have $\dim(G_q^r)=\dim(P_{q+1}^r)$. Referring to results of the previous subsection, we have $\dim(G_q^1)=q$, $\dim(G_q^2)=(q-1)q/2$, and $\dim(G_q^3)=(q-2)(q-1)q/6$.

The Helmholtz decomposition states that $U_q^r=G_q^r\oplus S_q^r$. Hence $\dim(S_q^r)=\dim(U_q^r)-\dim(G_q^r)$, i.e. $\dim(S_q^1)=1$, $\dim(S_q^2)=(q-1)(q+2)/2$ and $\dim(S_q^3)=(q-2)(q-1)(2q+3)/6$. The space of 1-form element is decomposed to

$$W_q^1=6\times(G_{q-1}^1\oplus S_q^1)\oplus 4\times(G_{q-1}^2\oplus S_q^2)\oplus(G_{q-1}^3\oplus S_q^3)$$

Consequently, the number of D.O.F. to be assigned on a p-simplex is $\dim(G_{q-1}^r)+\dim(S_q^r)$. Results are shown in Table II where the D.O.F. of $\dim(\tilde{G}_0)$ [equal to $4\times\dim(P_q^0)-\dim(\tilde{P}_0)=3$] are excluded from edge D.O.F. in the range space and added to the null space.

C. 2-Form Element (Facet Element)

The number of D.O.F. on facet and volume is determined in a similar way. Let us denote by

- V_q^r: the 2-form space of polynomials with degree up to q on S^r $(r=2,3)$ and having zero value on ∂S^r $(r=3)$ (2-vectors in S^r having zero normal component on ∂S^r).
- C_q^r: the subspace of V_q^r containing exact (closed) 2-forms (vectors which are curls of vector polynomial of degree $q+1$).

TABLE II
ASSIGNMENT OF D.O.F. OF 1-FORM (EDGE) ELEMENT

Spaces / D.O.F. on	$Z_q^1(S^3)=G_{q-1}$ (null space of d)	$Y_q^1(S^3)=S_q$ (range space of d)	$W_q^1(S^3)$ (function space)
Edges	6 × (q-1) + 3	6 × 1 − 3	6 × q
Facets	4 × (q-1)(q-2)/2	4 × (q-1)(q+2)/2	4 × (q-1)q
Volume	(q-1)(q-2)(q-3)/6	(q-1)(q-2)(2q+3)/6	(q-2)(q-1)q/2
Total	(q+1)(q+2)(q+3)/6-1	q(q+1)(2q+7)/6	q(q+2)(q+3)/2

- T_q^r: the complement of C_q^r in V_q^r (vectors having nonzero divergence).

We have $P_q(S^2)=V_q^2$, $(P_q(S^3))^3=4\times V_q^2\oplus V_q^3$ and hence, $\dim(V_q^2)=(q+1)(q+2)/2$, $\dim(V_q^3)=(q-1)(q+1)(q+2)/2$.

Since the curl operator is an isomorphism of S_{q+1}^r onto C_q^r, we have $\dim(C_q^r)=\dim(S_{q+1}^r)$, i.e. $\dim(C_q^2)=q(q+3)/2$ and $\dim(C_q^3)=(q-1)q(2q+5)/6$.

Otherwise, $V_q^r=C_q^r\oplus T_q^r$. Hence, $\dim(T_q^2)=1$ and $\dim(T_q^3)=q(q+1)(q+2)/6-1$.

According to the previous analysis, $W_q^2=4\times(C_{q-1}^2\oplus T_q^2)\oplus(C_{q-1}^3\oplus T_q^3)$. Consequently, the number of D.O.F. to be assigned on a r-simplex is $\dim(C_{q-1}^r)+\dim(T_q^r)$. Results are shown in Table III where the D.O.F. of $\dim(\tilde{C}_0)$ [equal to $\dim(S_q^1)-\dim(\tilde{G}_0)=3$] are excluded from facet D.O.F. in the range space and added to the null space.

In the case of $p=3$ (volume element), all D.O.F. are assigned to volume. No special analysis is needed.

Finally, the assignment of D.O.F. of p-form elements to r-simplex $(r=0,1,2,3)$ is summarized in Table IV.

The analysis of this section shows clearly the link of p-form elements on each simplex, in particular the link of null spaces of $p-1$-form element to the range spaces of p-form element under the differential operator. This analysis is helpful, not only for the generation of basis, but also to the application of gauge condition when necessary. For example, in the case of gauging a vector potential formulation, it shows clearly what is the null space to be removed.

The basis functions of p-form elements must be defined in such a way that the conformity and the unisolvence conditions are satisfied [4], i.e. each element must match the corresponding continuity condition of p-form field across the interface of elements, and the shape functions must be independent to provide an unique solution of the field equation. The following section gives the general expressions of the basis functions that fulfill the conformity requirement.

VI. GENERAL EXPRESSIONS OF BASIS FUNCTIONS

Let λ_i be barycentric coordinates of a point x with respect to a node i in S^3. Let $N_{(q-m)}=N_{(q-m)}(\lambda_i)$, $E_{(q-m)}=E_{(q-m)}(\lambda_i,\lambda_j)$, $F_{(q-m)}=F_{(q-m)}(\lambda_i,\lambda_j,\lambda_k)$ and $V_{(q-m)}=V_{(q-m)}(\lambda_i,\lambda_j,\lambda_k,\lambda_l)$, $1\le m\le 4$, respectively, polynomials of variables λ_i's with degree up to q–m. They will be further distinct with different superscript or/and subscript when used in different expressions.

TABLE III
ASSIGNMENT OF D.O.F. OF 2-FORM (FACET) ELEMENT

Spaces / D.O.F. on	$Z_q^2(S^3) = C_{q-1}$ (null space of d)	$Y_q^2(S^3) = T_q$ (range space of d)	$W_q^2(S^3)$ (function space)
Facets	4 × (q-1)(q+2)/2 + 3	4 ×1 - 3	4 × q(q+1)/2
Volume	(q-2)(q-1)(2q+3)/6	q(q+1)(q+2)/6 - 1	(q-1)q(q+1)/2
Total	q(q+1)(2q+7)/6	q(q+1)(q+2)/6	q(q+1)(q+3)/2

TABLE IV
ASSIGNMENT OF DEGREES OF FREEDOM

Element / D.O.F. on	0-form nodal	1-form edge	2-form facet	3-form volume
Vertices	4 × 1	-	-	-
Edges	6 × (q-1)	6 × q	-	-
Facets	4 × (q-2)(q-1)/2	4 × q(q-1)	4 × q(q+1)/2	-
Volume	$\frac{(q-3)(q-2)(q-1)}{6}$	$\frac{(q-2)(q-1)q}{2}$	$\frac{(q-1)q(q+1)}{2}$	$\frac{q(q+1)(q+2)}{6}$
Total D.O.F.	$\frac{(q+1)(q+2)(q+3)}{6}$	$\frac{q(q+2)(q+3)}{2}$	$\frac{q(q+1)(q+3)}{2}$	$\frac{q(q+1)(q+2)}{6}$

A. 0-Form Element (Nodal Element)

The basis functions of 0-form element W_q^0 take the following general expressions on each simplex:

$$\begin{aligned} N^n_{(q-1)}\lambda_i &\in P^0_q, & q&\geq 1;\\ E^n_{(q-2)}\lambda_i\lambda_j &\in P^1_q, & q&\geq 2;\\ F^n_{(q-3)}\lambda_i\lambda_j\lambda_k &\in P^2_q, & q&\geq 3;\\ V^n_{(q-4)}\lambda_i\lambda_j\lambda_k\lambda_l &\in P^3_q, & q&\geq 4. \end{aligned}$$

0-form element ensures C^0 continuity on the interface of adjacent elements. It belongs to the Hilbert space: $W_q^0(S^3) \subset H(\text{grad}) = \{v \in L^2(S^3)|\text{grad}v \in IL^2(S^3)\}$.

B. 1-Form Element (Edge Element)

We introduce first an operator $\mathcal{R}$, which rotates indices such that $\mathcal{R}^0 f_{ij} = f_{ij}\mathcal{R}^1 f_{ij} = f_{ji}$, and $\mathcal{R}^0 f_{ijk} = f_{ijk}\mathcal{R}^1 f_{ijk} = f_{jki}$, $\mathcal{R}^2 f_{ijk} = f_{kij}$, etc. The basis functions of W_q^1 on each simplex are:

$$\begin{aligned} \sum_{m=0}^{1} E^e_{m(q-1)}\mathcal{R}^m(\lambda_i\, d\lambda_j) &\in G^1_{q-1}\oplus S^1_q, & q&\geq 1;\\ \sum_{m=0}^{2} F^e_{m(q-2)}\mathcal{R}^m(\lambda_i\lambda_j\, d\lambda_k) &\in G^2_{q-1}\oplus S^2_q, & q&\geq 2;\\ \sum_{m=0}^{3} V^e_{m(q-3)}\mathcal{R}^m(\lambda_i\lambda_j\lambda_k\, d\lambda_l) &\in G^3_{q-1}\oplus S^3_q, & q&\geq 3; \end{aligned}$$

Since λ_i is a 0-form, the differential operator d can be read grad. This element satisfies the 1-form conformity (curl-conformal), i.e., $W_q^1(S^3) \subset H(\text{curl}) = \{\boldsymbol{u} \in IL^2(S^3)|$ curl $\boldsymbol{u} \in IL^2(S^3)\}$. The tangential components of the vector field are continuous on the interface of adjacent elements.

C. 2-Form Element (Facet Element)

Following functions spans the 2-form element W_q^2:

$$\begin{aligned} \sum_{m=0}^{2} F^f_{m(q-1)}\mathcal{R}^m(\lambda_i\, d\lambda_j\wedge d\lambda_k) &\in C^2_{q-1}\oplus T^2_q, & q&\geq 1;\\ \sum_{m=0}^{3} V^f_{m(q-2)}\mathcal{R}^m(\lambda_i\lambda_j\, d\lambda_k\wedge d\lambda_l) &\in C^3_{q-1}\oplus T^3_q, & q&\geq 2. \end{aligned}$$

This element is conforming in $H(\text{div}) = \{\boldsymbol{u} \in IL^2(S^3)$, div$\boldsymbol{u} \in L^2(S^3)\}$. The normal components of the vector field are continuous on the interface of adjacent elements.

D. 3-Form Element (Volume Element)

Finally, the basis functions of 3-form element are $\sum_{m=0}^{3} V^v_{m(q-1)}\mathcal{R}^m(\lambda_i\lambda_j\wedge d\lambda_k\wedge d\lambda_l) = V^v_{(q-1)} \in P_{q-1}, q\geq 1$. They are piece-wise continued functions [$(q-1)$ order polynomials].

The coefficients of polynomials in the above expressions must be determined in order that the functions are linearly independent and model correctly the null and range spaces of the differential operator on each simplex. The coefficients can

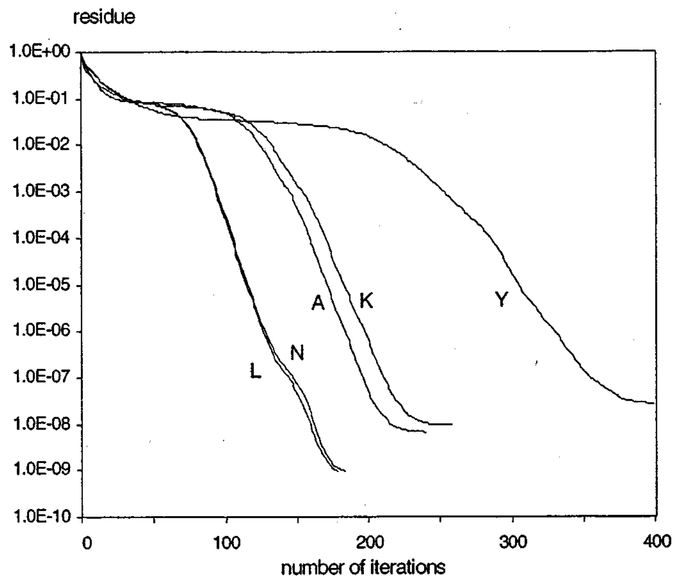

Fig. 3. Comparison of convergence behaviors of various elements.

be determined in various ways and this leads to different types of elements [5]–[10].

VII. Comparison of Some 2nd Order 1-Form Elements

In the case of 2nd order 1-form element, we assign 2 D.O.F. per edge and 2 D.O.F. per facet. Different types of elements have been developed [5], [8]–[10]. We have applied them in a formulation in terms of magnetic vector potential to solve magnetostatic problems [13] and in a combined magnetic vector potential and electric scalar potential formulation to compute eddy currents [14].

In the magnetostatic application [13], we have shown that, the field distributions obtained with all elements are nearly identical. However, the conditioning of the matrix system, and hence the convergence behaviors are very different. The best convergence is obtained by Lee's element [5]. In addition Lee' element has the advantage of being hierarchical [6].

When applying these elements to solve eddy current problems, Lee's element converges still more quickly than others do. However, Lee' element suffers the asymmetry problem of the facet basis functions: choosing differently two facet basis functions among three available leads to different modeling spaces and has a harmful influence on numerical results. In order to surmount this drawback but still keep the hierarchical property, we introduced a new basis [14] in modifying the facet basis functions. This basis stays intact with the random choice of facet basis functions, and keeps the good conditioning of the matrix system and the hierarchical property. A comparison with other elements in eddy current application is shown in [14].

This element is also applied for solving magnetostatic problems. In the case of a magnetic circuit problem (see definition of the problem in [13]), the field distribution is almost identical to that of other elements. The convergence behavior is shown in Fig. 3, where the curves L, A, K, Y and N correspond, respectively, to the elements given in [5], [8]–[10] and our proposed one [14]. It can be seen that our element has a good convergence behavior as Lee's.

These comparative studies show that, once we model correctly the null and range space of the curl operator, all elements provide same accuracy of results. But the conditioning of their matrix system is very different. So the conditioning of the system must be taken as one of the important criteria when constructing element bases.

VIII. Procedure for the Generation of Bases

The above analysis showed the link between differential form-based elements of different degrees and of different order. This link illustrates the inclusion property of p-form elements and is helpful for the generation of their bases.

Taking independent functions described in Section VI will form the interpolatory basis. However, recent studies showed the advantages of hierarchical bases [6]. The hierarchy means that the basis functions of the high order elements include all basis functions of the spaces of lower order elements. This property allows mixing of different order of elements in the same mesh without the difficulty of matching field continuities. It is helpful for mixed h- and p-version adaptive mesh generation or for the development of adaptive multigrid solvers [15]–[16]. Since the spaces of higher order elements include those of lower order elements, the generation of hierarchical bases can be conveniently realized.

For example, the hierarchical basis of 1-form element can be generated using the following procedure. Supposing the 0-form element has been generated with a hierarchical basis [17] and let the first order 1-form element W_1^1 be the Whitney edge element. We denote by $\hat{P}_q^r = P_q^r \backslash P_{q-1}^r$ and $\hat{G}_{q-1}^r = G_{q-1}^r \backslash G_{q-2}^r$, $\hat{S}_q^r = S_q^r \backslash S_{q-1}^r$, respectively, spaces of basis functions over the r-simplex of 0-form and 1-form elements when the basis is upgraded from the order $q-1$ to q. $\hat{E}_q^n \in \hat{P}_q^1$, $\hat{F}_q^n \in \hat{P}_q^2$, $\hat{V}_q^n \in \hat{P}_q^3$ are hierarchical basis functions of 0-form element generated on edge, facet and volume, when the order is upgraded from $q-1$ to q.

A general procedure for the generation of hierarchical basis of q-order 1-form element is, for $s = 2$ to $s = q$,

- Taking 1 function which is differential (gradient) of $\hat{E}_s^n \in \hat{P}_s^1$ on each edge to generate $\hat{G}_{s-1}^1$.
- Taking $s-2$ functions, which are differential of $\hat{F}_s^n \in \hat{P}_s^2$ on each facet to generate $\hat{G}_{s-1}^2$.
- Adding s functions of non zero curl $\sum_{m=0}^{2} \tilde{F}_{m(s-2)}^e \mathcal{R}^m(\lambda_i \lambda_j \, d\lambda_k) \in \hat{S}_s^2$ on each facet to complete the facet basis.
- Taking $(s-2)(s-3)/2 (s \geq 3)$ functions which are differential of $\hat{V}_s^n \in \hat{P}_s^3$ to generate $\hat{G}_{s-1}^3$.
- Adding $s(s-2)(s \geq 3)$ nonzero curl functions $\sum_{m=0}^{3} \tilde{V}_{m(s-3)}^e \mathcal{R}^m(\lambda_i \lambda_j \lambda_k \, d\lambda_l) \in \hat{S}_s^3$ to complete the volume basis.

A basis of 1-form element of the order up to 3 generated with this procedure is shown in Table V. The number before the function indicates the number that take the same form of the basis function in rotating indices on a simplex. Application of this basis in the case of 2nd order is given in [14] and reported in the previous section.

A good element basis must lead to a good conditioning of the matrix system. The coefficients of polynomials in the basis

TABLE V
A HIERARCHICAL 1-FORM Q-ORDER (UP TO 3) ELEMENT BASIS

q			edge functions		Facet functions		Volume functions	
		1	1	$\lambda_i d\lambda_j - \lambda_j d\lambda_i$		-		-
	2		1	$d(\lambda_i\lambda_j)$	2	$\lambda_i(\lambda_j d\lambda_k - \lambda_k d\lambda_j)$		-
3			1	$d(\lambda_i\lambda_j^2 - \lambda_j\lambda_i^2)$	1 3	$d(\lambda_i\lambda_j\lambda_k)$ $\lambda_i^2(\lambda_j d\lambda_k - \lambda_k d\lambda_j)$	3	$\lambda_i\lambda_j(\lambda_k d\lambda_l - \lambda_l d\lambda_k)$

functions must be determined in order that the matrix conditioning is optimal. A good matrix conditioning requires the matrix to be diagonally dominant. The ideal case is to have the basis functions mutually orthogonal. However in practice, it is almost impossible to get such an ideal basis. What we can try is to make them as orthogonal as possible. It should be pointed out that the orthogonality is problem dependent, i.e. the shape of elements, the material property of the problems. It is difficult to provide a general rule to realize an orthogonal basis and further investigation has to be carried out.

IX. CONCLUSIONS

With the help of De Rham's complex, the null spaces and the range spaces of the differential operator of p-form elements as well as their link are clearly shown. After a complete analysis of the assignment of D.O.F. on each simplex, the general expressions of p-form elements are given. The determination of their coefficients varies and this leads to different kind of elements. Comparison of several 2nd order 1-form elements shows that even though they provide the same accuracy of results, their matrix conditioning is very different. Utilizing the inclusion property of the spaces of p-form elements of different degrees and orders, a procedure for the generation of the hierarchical basis is given. Determination of coefficients in order to have an optimal matrix conditioning needs further investigation.

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