Solving 3D Eddy Current Problems Using Second Order Nodal and Edge Elements

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***Abstract*—Several 2nd order nodal and edge elements have been applied in a potential formulation to solve 3D eddy current problems. The asymmetry of the facet related functions in the edge element basis is discussed. A new basis is proposed. Application of a gauge condition for the uniqueness of vector potential is cumbersome in the case of high order elements. This work shows that the system converges without explicit gauge condition when using the bi-conjugate gradient method. The performance of different elements is compared through an example.**

***Index Terms*—Eddy currents, finite element modeling, second order edge elements.**

I. INTRODUCTION

THE WHITNEY (nodal, edge, facet, and volume) elements have proven their efficiency in electromagnetic field computation in the last decade [1]. They belong to differential forms of different degrees. The Whitney edge element (1-form element) has been widely used for solving electromagnetic field problems in various frequency ranges. However, these elements are built in first order.

The theory of high order edge (curl-conformal) and facet (div-conformal) elements was advanced in the early of 80's in [2]. Unfortunately, in this reference, no specific vector basis function was reported. Further investigation has been carried out in recent years by different researchers. Different high order edge elements were developed [3]–[7]. These are mostly applied in the high frequency domain. Few works can be found in low frequency and static field applications. The main difficulty in low frequency applications seems to be the application of gauge conditions.

This paper investigates some 2nd order edge elements in the computation of eddy currents using a potential formulation. We will show that, the system converges without explicit gauge condition. This is the same conclusion as in the case of first order elements.

One of the difficulties in the application of 2nd order edge elements is the asymmetry of the facet related basis functions. This issue will be discussed. A new basis preserving the hierarchical property and getting rid of the asymmetry problem of the basis function will be proposed. The performance (accuracy and convergence behavior) of different elements is then compared.

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II. SECOND ORDER EDGE ELEMENTS

The elements considered in this paper are tetrahedral. The second order nodal element is the Lagrange type and contains 10 nodes (vertices plus one node in the middle of each edge). The high order edge elements must model correctly the range space and the null space of the curl operator. In the case of 2nd order edge elements, the curl field must be complete to the first order in the range of the curl operator. The number of degrees of freedom needed to model a first order vector field is 12. The divergence free condition reduces this number to 11. To model the null space of the curl operator (the gradient field), the number of degrees of freedom is 9. In consequence, the number of degrees of freedom required in a 2nd order tetrahedral edge element is 20. These degrees of freedom are commonly assigned to the edges and facets (2 per edge and 2 per facet).

The basis functions related to the edges and the facets take the following general forms:

On an edge defined by the vertices ij

$$w_{ij} = (a_1 + b_1\lambda_i + c_1\lambda_j)\lambda_i\nabla\lambda_j + (a_2 + b_2\lambda_j + c_2\lambda_i)\lambda_j\nabla\lambda_i, \tag{1.a}$$

where λ_i is the barycentric coordinate of a point with respect to the vertex i. Permuting the indices ij in this expression gives another basis function defined on the same edge.

On a facet defined by vertices ijk:

$$w_{ijk} = d_1\lambda_i\lambda_j\nabla\lambda_k + d_2\lambda_j\lambda_k\nabla\lambda_i + d_3\lambda_k\lambda_i\nabla\lambda_j. \tag{1.b}$$

Rotating indices ijk leads to three functions, but only two of them are used. Let us introduce a rotation operator, noted by $\mathcal{R}$, such that $\mathcal{R}^0 f_{ijk} = f_{ijk}$, $\mathcal{R}^1 f_{ijk} = f_{jki}$, and $\mathcal{R}^2 f_{ijk} = f_{kij}$. The second facet related function might be $\mathcal{R}w_{ijk}$, or $\mathcal{R}^2 w_{ijk}$.

Let W_2^1 denote the space of second order edge element defined by (1.a) and (1.b). It can be shown that W_2^1 belong to the following domain of the curl operator:

$$W_2^1 \subset H(\text{curl}) = \{\boldsymbol{u} | \boldsymbol{u} \in IL^2(\Omega),\ \text{curl}\,\boldsymbol{u} \in IP_1(\Omega) \cap D(\Omega)\}$$

where

- $IL^2(\Omega)$ is the Hilbert space of a square integrable vector field,
- $IP_1(\Omega)$ the three dimensional space of first order polynomials and
- $D(\Omega)$ the space of divergence free functions, over Ω, respectively.

Each of the function (1.a) and (1.b) is tangentially continuous through the interface of two adjacent elements. They describe a complete first order curl field over a tetrahedron.

III. Eddy Current Formulation

Consider an eddy current problem in a bounded region Ω, which includes a conducting region Ω_c and an excitation coil Ω_j, carrying a current $\boldsymbol{j}_0$. The boundary of Ω is split in two: $\partial\Omega = \Gamma_e \cup \Gamma_h$ and the intersection of Γ_e and Γ_h is empty. On the boundary, the boundary conditions $\boldsymbol{n} \times \boldsymbol{e} = 0$ on Γ_e and $n \times h = 0$ on Γ_h hold.

To solve this problem we use a formulation in terms of magnetic vector potential $\boldsymbol{a}$ and the time integral of electric scalar potential ψ. Solving weakly Ampere's theorem, yields: find $\boldsymbol{a} \in W_{2e}^1$ and $\psi \in W_{2e}^0$ such that

$$\int_\Omega \frac{1}{\mu} \operatorname{curl} \boldsymbol{a}' \cdot \operatorname{curl} \boldsymbol{a} \, d\Omega + \frac{d}{dt} \int_\Omega \sigma \boldsymbol{a}' \cdot \boldsymbol{a} \, d\Omega + \frac{d}{dt} \int_\Omega \sigma \boldsymbol{a}' \cdot \operatorname{grad} \psi \, d\Omega + \int_{\Omega_t} \operatorname{curl} \boldsymbol{a}' \cdot \boldsymbol{t}_0 \, d\Omega = 0 \qquad \forall \boldsymbol{a}' \in W_{2\,e}^1 \tag{2.a}$$

$$\frac{d}{dt} \int_{\Omega_c} \sigma \operatorname{grad}\psi' \cdot \boldsymbol{a} \, d\Omega + \frac{d}{dt} \int_{\Omega_c} \sigma \operatorname{grad} \psi' \cdot \operatorname{grad}\psi \, d\Omega = 0 \quad \forall \psi' \in W_{2\,e}^0 \tag{2.b}$$

where

$$W_{2e}^1 = \{\boldsymbol{a} \in W_2^1 | \boldsymbol{n} \times \boldsymbol{a} = 0 \text{ on } \Gamma_e\},$$
$$W_{2e}^0 = \{\psi \in W_2^0 | \psi = 0 \text{ on } \Gamma_e\},$$

and $\boldsymbol{t}_0$ is a vector potential such that $\boldsymbol{j}_0 = \operatorname{curl} \boldsymbol{t}_0$, introduced to enforce $\operatorname{div} \boldsymbol{j}_0 = 0$. It is defined in a domain Ω_t containing the excitation coil Ω_j. W_2^1 is the previously described edge element space and W_2^0 is the space of the common 2nd order Lagrange type nodal elements.

In the above equation, the system is singular in both conducting and nonconducting regions because W_2^1 includes the null space of the curl operator and hence the gradient space of nodal elements. The solution of $\boldsymbol{a}$ is not unique and a gauge condition must be applied to ensure its uniqueness. The number of redundant unknowns to be removed is the dimension of the null space of the curl operator. A similar technique like the tree gauge in the case of 1st order element [8] can be extended to the case of high order elements. However, the construction of a tree in the case of high order element can be very complicated. It requires a careful analysis of the null space of the curl operator of the edge elements [9], [10].

According to the experience with 1st order edge element, the use of a tree can cause ill conditioning of the system and decrease the accuracy of the solution [11]. The tree technique seems not to be the best solution. It has been shown that, when using an iterative solver like the conjugate gradient method, the system is implicitly gauged by the solver itself and the convergence behaves much better than the gauged formulation. In our application, the singular system will be solved using a conjugate gradient type iterative solver and no explicit gauge condition will be applied.

IV. Description of Several 2nd Order Elements

A. Some Kinds of 2nd Order Edge Elements

The coefficients in (1) can be determined in various ways and this leads to different kinds of elements [3]–[7]. Some of these elements are considered in our study. Their edge and facet related basis functions are given below.

1) Lee's element [3]

$$w_{ij} = \lambda_i \nabla \lambda_j, \tag{3.a}$$
$$w_{ijk} = \lambda_i \lambda_j \nabla \lambda_k \tag{3.b}$$

2) Ahagon's element [6]

$$w_{ij} = \lambda_i(-1 + 4\lambda_i)\nabla\lambda_j + \lambda_j(1 - 4\lambda_i)\nabla\lambda_i, \tag{4.a}$$
$$w_{ijk} = 4\lambda_i\lambda_j\nabla\lambda_k - 4\lambda_j\lambda_k\nabla\lambda_i \tag{4.b}$$

3) Yioultsis's element [7]

$$w_{ij} = \lambda_i(-4 + 8\lambda_i)\nabla\lambda_j + \lambda_j(2 - 8\lambda_i)\nabla\lambda_i, \tag{5.a}$$
$$w_{ijk} = 16\lambda_i\lambda_j\nabla\lambda_k - 8\lambda_j\lambda_k\nabla\lambda_i - 8\lambda_k\lambda_i\nabla\lambda_j \tag{5.b}$$

It is noted that Lee's element includes the 1st order edge element basis. It belongs to the Webb's hierarchical elements [4]. The hierarchy means that the basis functions of the high order elements include all basis functions of the spaces of lower order elements. This property allows mixing of different order of elements in the same mesh without the difficulty of matching field continuities. This is a helpful property for mixed h- and p-version adaptive mesh generation or for using adaptive multigrid solvers.

B. On the Asymmetry of Facet Related Basis Functions

The 2nd order edge element assigns two degrees of freedom to each facet. However, rotating the indices ijk of the facet related basis function (1.b) gives three functions. The basis is hence asymmetric.

This asymmetry may cause, first, difficulties in the numerical implementation. Special attention must be paid to choose the same facet basis functions for the two adjacent tetrahedra. In order to avoid ambiguity, in our application, the two basis functions w_{ijk} on a facet ijk are chosen such that $i < j$ and $i < k$, ensuring a unique choice of degrees of freedom on the facets.

Second, we may ask if the random choice of the asymmetric facet functions will influence the accuracy of results. To check this point, let us see the dependence of those three functions. Using again the rotation operator $\mathcal{R}$, we note that Ahagon's and Youlsis' elements satisfy the relation:

$$\sum_{r=0}^{2} \mathcal{R}^r w_{ijk} = 0 \tag{6}$$

This means that taking any two of three functions spans the same space, and hence we can expect that the choice of any two of three functions will not influence the numerical results. Instead, Lee's element does not fulfill this relation, but $\sum_{r=0}^{2} \mathcal{R}^r w_{ijk} = \nabla(\lambda_i\lambda_j\lambda_k)$. This means different choices differ from a gradient

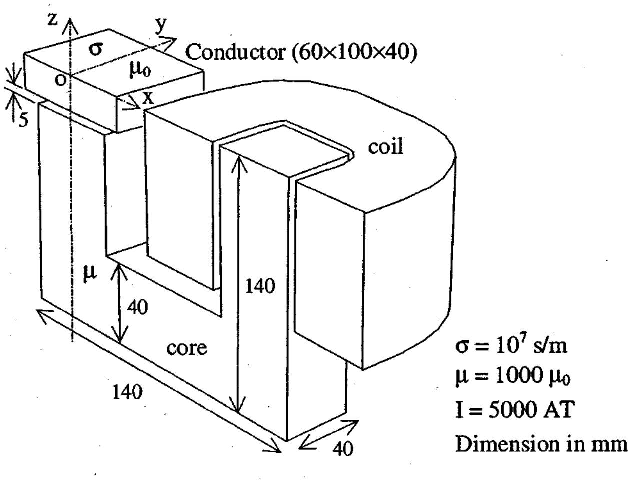


Fig. 1. Example of an eddy current problem

field and hence lead to different modeling spaces. The numerical results will depend on the choice of facet basis functions. Nevertheless, it should be noted that even though Lee's functions themselves do not satisfy (6), their curls do. This means the curl field will not depend on the choice of bases. This is the case when solving magnetostatic problems [12].

To get a symmetric edge element, Kameari proposed to add one node in the middle of each facet [13]. The terms $\lambda_i\lambda_j\lambda_k$ are added to the second order polynomials to form nodal basis functions. This results in a 14 nodes nodal element. To build 2nd order edge elements, 3 degrees of freedom are assigned on each face so that the functions are symmetric. The total number of degrees of freedom becomes 24. Adding 4 nodes in the element increases the dimension of the null space of the curl operator to 13 but doesn't affect the dimension of its range space. It does not improve the accuracy of the curl field [12]. Since the number of unknowns becomes much lager, it is not considered in this paper.

C. Proposal of a New Basis

Lee' element has the advantage of being hierarchical but suffers the problem of asymmetry for the facet basis functions. To get rid of this problem, we propose to modify the facet functions and give the new basis below:

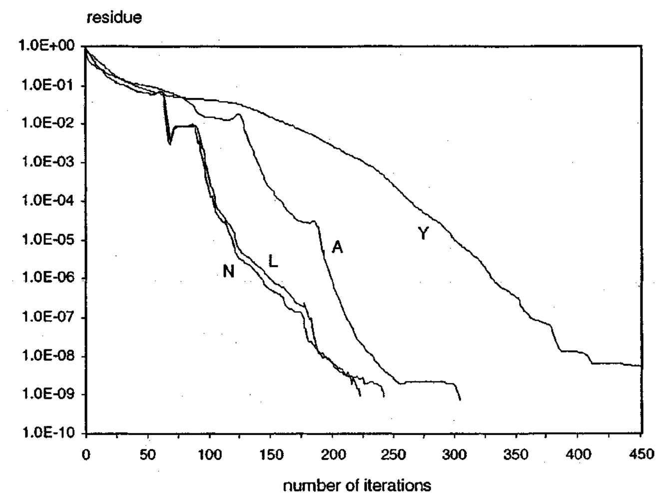
$$w_{ij} = \lambda_i \nabla \lambda_j, \tag{7.a}$$

$$w_{ijk} = \lambda_i\lambda_j \nabla \lambda_k - \lambda_j\lambda_k \nabla \lambda_i \tag{7.b}$$

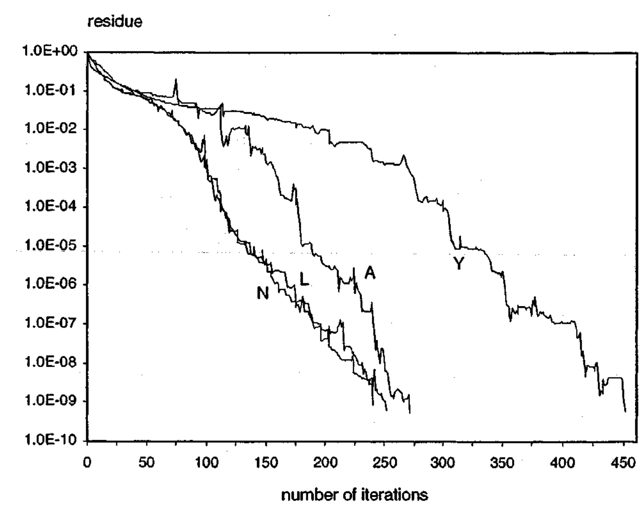
This basis includes the 1st order edge element like Lee's, and hence we keep the hierarchy property. In addition, the face functions satisfy (6) and the modeling space will not be influenced by the random choice of facet basis functions.

V. Comparison of Results

The example to be considered concerns a conductor inserted in the Centre of the air gap of a magnetic circuit. One fourth of the domain is shown in Fig. 1. The study domain is meshed by 1600 tetrahedral elements. The edge elements are applied on the whole region whereas the nodal elements only in the conducting domain. There are 9942 unknowns related to edge elements, of which 3634 are associated with edges and 6318 with facets. The degrees of freedom related to nodal elements are 486. The excitation current in the coil is sinusoidal. The equation (2) is complex and solved with a diagonal preconditioned bi-conjugate gradient method.



(a)

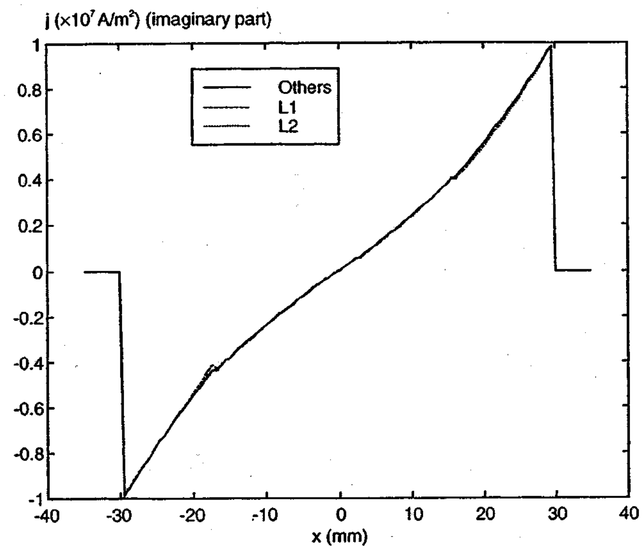


(b)

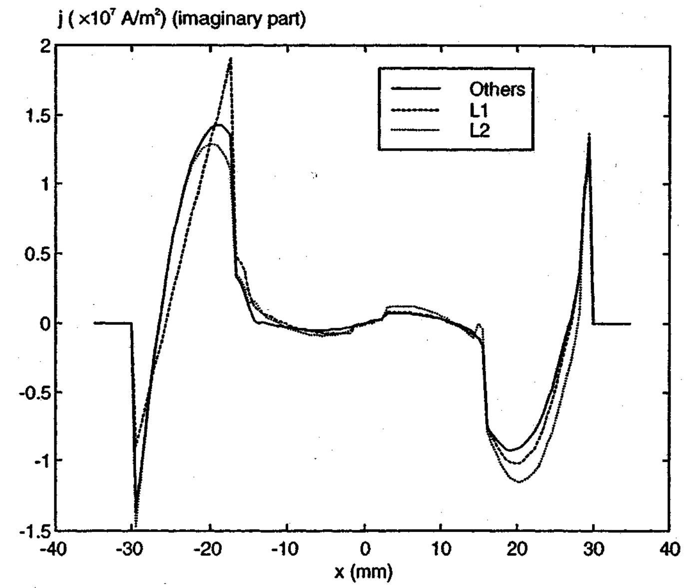
Fig. 2. Convergence behavior of different elements (a) $f = 50$ Hz, (b) $f = 1$ kHz.

The conditioning of the complex system will depend on the frequency, i.e. the ratio skin depth (δ) over mesh size (h). In order to check the behavior of these elements under different ratios δ/h, the problem is solved under two frequencies 50 Hz and 1 kHz with the same mesh. In the conducting region, the average mesh size is about 10 mm. At 50 Hz, the ratio $\delta/h \approx 2.2$; and at 1 kHz, $\delta/h \approx 0.5$.

The elements (3)–(5) and (7) are used to solve this problem. No specific gauge condition is applied to ensure uniqueness of the vector potential. The convergence behaviors are compared in Fig. 2. for both frequencies. It is observed that in both cases, the best convergence is obtained for Lee's element and the new proposed elements (curves L and N). The convergence of Yioultsis's element is relatively slow (curve Y). Ahagon's element (curve A) has a rate of convergence faster than Y's



(a)



(b)

Fig. 3. y-component current density on the center line ($y = 0, z = 0$) of the conductor (a) $f = 50$ Hz, (b) $f = 1$ kHz.

but slower than L's. This is the same conclusion as we observed for the magnetostatic case [12]. It should be noted that, in the conducting region, at low frequency $(\delta/h > 1)$, the magnetic energy term dominates, the conditioning of the system is mostly determined by the eigenvalues of the curl-curl matrix (real part of the complex system). On the other hand, at high frequency $(\delta/h < 1)$, the joule losses term becomes more important and the conditioning of the system is mostly determined by the eigenvalues of the imaginary part. We concluded that, in both cases, Lee's and the new proposed elements give the best conditioning of the system.

We now examine the numerical results. In particular, we will check if the random choice of facet related basis functions (the asymmetric basis) influences the results. To do this, for each kind of element, we considered two bases. One consists in choosing two basis functions w_{ijk} such that $i < j$ and $i < k$, as previously indicated. The other takes two functions w_{ijk} such that $i > j$ and $i > k$. The curves plotted in Fig. 3 correspond to the y-component of the current density (imaginary part) along a line in the middle of the conductor, obtained with different elements. We observed that, all elements give the same field results (the difference is smaller than 0.1 percent) except Lee's. The choice of facet basis functions has no influence on the numerical results for Yioultsis's, Ahagon's and the new proposed elements. Instead, Lee's element suffers the problem of the asymmetry. Using two bases, we get two different results. The difference is small in the case of big δ/h ratios. Because in this case, the curl-curl term dominates and the curl of Lee's basis satisfies the dependence relation (6). However, when the frequency increases (the ratio δ/h diminishes), it is the term of the edge element function itself that becomes dominant. The difference of results becomes bigger as we can observe from Fig. 3(b). We conclude that the accuracy of Lee's elements is affected by the random choice of facet functions.

VI. Conclusions

Several 2nd order edge elements have been applied in a magnetic vector and electric scalar potential formulation to solve eddy current problems. Results show that the convergence of the system is achieved without explicit gauge condition when using a conjugate gradient method.

Through a comparison on an eddy current problem using the same mesh under different frequencies, we conclude that the conditionings of the matrix system of these elements are very different. This is clearly illustrated by their convergence behavior. The systems of Lee's element and the new proposed element are better conditioned than the others.

As for the numerical results, all these elements provide nearly the same field and current distribution as long as the skin depth/mesh size ratio is large (larger than 2). When this ratio diminishes, the results with Lee's element are influenced by the random choice of asymmetric facet basis functions while the other elements (including the new proposed element) stay intact with the random choice.

In addition, with the new proposed element basis, we keep the hierarchical property of Lee's element. This property is helpful for coupling with boundary element methods that use elements of different order, for mixed h- and p-version adaptive mesh refinement, and for solving problems using adaptive multigrid methods.

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