Computation of magnetostatic field using second order edge elements in 3D

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Keywords *Edge elements, Finite element modeling, Magnetostatics*

Abstract *Several second order edge elements have been applied to solving magnetostatic problems. The performances of these elements are compared through an example of magnetic circuit. In order to ensure the compatibility of the system equations and hence the convergence, the current density is represented by the curl of a source field. This avoids an explicit gauge condition which is cumbersome in the case of high order elements.*

I. Introduction

Improvement of the accuracy in finite element modeling can be achieved through two methods: local mesh refinement and increase of the order of the shape functions of the elements. Local mesh refinement leads in some cases to deformed elements, which may worsen the stability of the system and the accuracy of the results. The use of high order elements turns out to be more effective in such situations.

The Whitney (nodal, edge, facet and volume) elements have proven their efficiency in electromagnetic field computation in the last decade (Bossavit, 1988). They are differential forms of different degrees. The main properties of these elements are: conformity (matching the corresponding field continuity conditions) and inclusion (the element of low degree is included in the element of high degree). The Whitney edge element (one-form element) has been widely used for solving electromagnetic field problems in various frequency ranges. However, these elements are built in first order.

The theory of high order edge (curl-conformal) and facet (div-conformal) elements was advanced in the early 1980s in Nédélec (1980). Unfortunately, in this reference, no specific vector basis function was reported. Further investigation has been carried out in recent years by different researchers.

Different high order edge elements were developed (Lee *et al.*, 1991; Webb and Forghani, 1993; Wang and Ida, 1993; Ahagon and Kashimoto, 1995; Yioultsis and Tsiboukis, 1996; Kameari, 1998). These are mostly applied in the high frequency domain. Only few works can be found in low frequency and

COMPEL – The International Journal for Computation and Mathematics in Electrical and Electronic Engineering, Vol. 18 No. 3, 1999, pp. 361-371.

static field applications. The main difficulty in low frequency applications seems to be the application of gauge conditions.

This paper investigates several second order edge elements in the computation of magnetostatic fields. We will show that, in the case of a compatible formulation, when using an iterative solver such as the conjugate gradient method, the system converges without an explicit gauge condition. This is the same conclusion as in the case of first order element. The performance (accuracy and convergence behavior) of different elements is then compared.

II. Different types of second order edge elements

In this paper we consider the case of tetrahedral elements. The second order nodal element built on the tetrahedron is the Lagrange type and contains ten nodes (vertices plus one node in the middle of each edge). The high order edge elements must model correctly the range space and the null space of the curl operator. In the case of second order edge elements, the curl field must be complete to the first order in the range of the curl operator. The number of degrees of freedom to model a first order vector field is 12. The divergence free condition reduces this number to 11. To model the null space of the curl operator (the gradient field), the number of degrees of freedom is nine. In consequence, the number of degrees of freedom required in second order tetrahedral edge element is 20. These degrees of freedom are commonly assigned to the edges and facets (two per edge and two per facet).

The basis functions related to the edges and the facets take the following general forms:

(1) On an edge defined by the vertices {i, j}

$$w_{ij} = \lambda_i(a_1 + b_1\lambda_i + c_1\lambda_j)\Delta\lambda_j + \lambda_j(a_2 + b_2\lambda_j + c_2\lambda_i)\nabla\lambda_i, \quad (1.a)$$

where λ_i is the barycentric coordinate of a point with respect to the vertex i. Permuting the indices ij in this expression gives another base function defined on the same edge.

(2) On a facet defined by vertices {i, j, k}:

$$w_{ijk} = d_1\lambda_i\lambda_j\nabla\lambda_k + d_2\lambda_j\lambda_k\nabla\lambda_i + d_3\lambda_k\lambda_i\nabla\lambda_j. \quad (1.b)$$

Rotating indices ijk leads to three basis functions on the surface, but only two of them are used.

Let W_2^1 denote the space of second order edge element defined by (1.a) and (1.b). It can be shown that W_2^1 belongs to the following domain of the curl operator:

$$W_2^1 \subset H(\mathrm{curl}) = \{\mathbf{u}|\mathbf{u} \in IL^2(\Omega), \mathrm{curl}\,\mathbf{u} \in IP_1(\Omega) \cap D(\Omega)\}$$

where $IL^2(\Omega)$ is the Hilbert space of a square integrable vector field, $IP_1(\Omega)$ the three dimensional space of first order polynomials and $D(\Omega)$ = ker(div) the space of divergence free functions, over Ω, respectively.

Each of the functions (1.a) and (1.b) is tangentially continuous through the interface of two adjacent elements. In general, w_{ij} forms a second order vector field turning around the opposite edge. Its circulation vanishes on all edges except on the edge ij. Each term of the function w_{ijk} describes a second order vector field normal to the facet opposite the vertex i, j, or k. Consequently, the function w_{ijk} forms a field turning around edges that do not belong to the surface ijk. The circulation of w_{ijk} is obviously zero along all edges.

The element defined by (1.a) and (1.b) describes a complete first order curl field:

$$\text{curl } w_{ij} = [a_1 - a_2 + (2b_1 - c_2)\lambda_i + (-2b_2 + c_1)\lambda_j]\nabla\lambda_i \times \nabla\lambda_j, \qquad (1.c)$$

$$\begin{aligned}\text{curl } w_{ijk} =& (d_1 - d_3)\lambda_i\nabla\lambda_j \times \nabla\lambda_k + (d_2 - d_1)\lambda_j\nabla\lambda_k \times \nabla\lambda_i \\ &+ (d_3 - d_2)\lambda_k\nabla\lambda_i \times \nabla\lambda_j,\end{aligned} \qquad (1.d)$$

provided that the coefficients of the first order terms in these expressions are not simultaneously zero.

It must be emphasized that, in general, the degrees of freedom do not have a direct physical meaning such as the circulation of a field along edges, unless an orthogonal condition (the line integrals of the basis function on each edge is independent from each other) is satisfied. Unfortunately, this condition cannot be realized because the line integrals of w_{ij} and w_{ijk} on a line given on the facet ijk are usually not independent. But in general case, the circulation of a field along an edge is a linear combination of several degrees of freedom.

The coefficients in expressions (1.a) and (1.b) can be determined in various ways and this leads to different kinds of elements.

A. Lee's element

The basis functions that related on the edges and the facets of Lee's element (Lee *et al.*, 1991) are, respectively,

$$w_{ij} = \lambda_i\nabla\lambda_j, \qquad (2.a)$$

$$w_{ijk} = \lambda_i\lambda_j\nabla\lambda_k, \qquad (2.b)$$

The curls of Lee's element are

$$\text{curl } w_{ij} = \nabla\lambda_i \times \nabla\lambda_j, \qquad (2.c)$$

$$\text{curl } w_{ijk} = \lambda_i\nabla\lambda_j \times \nabla\lambda_k - \lambda_j\nabla\lambda_k \times \nabla\lambda_i \qquad (2.d)$$

It can be noted that Lee's element belongs to the Webb's hierarchical elements (Webb and Forghani, 1993). The hierarchy means that the basis functions of the high order elements include all basis functions of the spaces of lower order elements. This allows mixing of different order of elements in the same mesh without the difficulty of matching field continuities. It is a helpful property for adaptive mesh (mixed *h*- and *p*-refinement) generation.

B. Ahagon's element

Ahagon's element (Ahagon and Kashimoto, 1995) is derived from the decomposition of the gradient of second order nodal shape functions. The inclusion property requires that the gradient of second order nodal element is included in the second order edge element. This property means that the sum of edge element basis functions for the edges meeting at a vertex must be the gradient of nodal function on this vertex:

$$\sum_{j} w_{ij} = \text{grad } w_i,$$

where w_i is the basis function of the second order nodal element related to node i. The basis function derived in such a manner has the following form:

$$w_{ij} = \lambda_i(-1 + 4\lambda_i)\nabla\lambda_j + \lambda_j(1 - 4\lambda_i)\nabla\lambda_i, \tag{3.a}$$

$$w_{ijk} = 4\lambda_i\lambda_j\nabla\lambda_k - 4\lambda_j\lambda_k\nabla\lambda_i \tag{3.b}$$

Taking the curl, we have

$$\text{curl } w_{ij} = (-2 + 12\lambda_i)\nabla\lambda_i \times \nabla\lambda_j, \tag{3.c}$$

$$\text{curl } w_{ijk} = 4\lambda_i\nabla\lambda_j \times \nabla\lambda_k - 8\lambda_j\nabla\lambda_k \times \nabla\lambda_i + 4\lambda_k\nabla\lambda_i \times \nabla\lambda_j \tag{3.d}$$

C. Yioultsis' element

Yioultsis' element (Yioultsis and Tsiboukis, 1996) takes the weighted fields as degrees of freedom. By applying some constraints such that the linear combination of basis functions of edge elements gives the gradient of nodal elements, the following basis functions are achieved:

$$w_{ij} = \lambda_i(-4 + 8\lambda_i)\nabla\lambda_j + \lambda_j(2 - 8\lambda_i)\nabla\lambda_i, \tag{4.a}$$

$$w_{ijk} = 16\lambda_i\lambda_j\nabla\lambda_k - 8\lambda_j\lambda_k\nabla\lambda_i - 8\lambda_k\lambda_i\nabla\lambda_j \tag{4.b}$$

Their curls are

$$\text{curl } w_{ij} = 6(-1 + 4\lambda_i)\nabla\lambda_i \times \nabla\lambda_j, \tag{4.c}$$

$$\text{curl } w_{ijk} = 24(\lambda_i\nabla\lambda_j \times \nabla\lambda_k - \lambda_j\nabla\lambda_k \times \nabla\lambda_i) \tag{4.d}$$

D. Kameari's element

The placement of degrees of freedom on the facets in above described elements is asymmetric. This may cause some difficulty for the numerical implementation. To get a symmetric edge element, Kameari proposed to add one node in the middle of each facet (Kameari, 1998). This results in a 14 nodes nodal element. The terms $\lambda_i\lambda_j\lambda_k$ are added to the second order polynomials to form nodal basis functions. To build second order edge element, three degrees of freedom are assigned on each face. The total number of degrees of freedom is 24. By doing so and after applying an orthogonal condition such that

$$\int_{l_j} \mathbf{w}_i d\mathbf{l} = \delta_{ij}$$

where δ_{ij} is the Kronecker, the following basis functions are obtained:

$$\begin{aligned} w_{ij} =& [\lambda_i(-33 + 63\lambda_i + 30\lambda_j)\nabla\lambda_j \\ &+ \lambda_j(-5 + 15b_2\lambda_j - 18c_2\lambda_i)\nabla\lambda_i]/10, \end{aligned} \tag{5.a}$$

$$w_{ijk} = 3(31\lambda_i\lambda_j\nabla\lambda_k + 7\lambda_j\lambda_k\nabla\lambda_i + 7\lambda_k\lambda_i\nabla\lambda_j)/5 \tag{5.b}$$

The curls of these functions are

$$\text{curl } w_{ij} = 2(-7 + 36\lambda_i)\nabla\lambda_i \times \nabla\lambda_j/5, \tag{5.c}$$

$$\text{curl } w_{ijk} = 72(\lambda_i\nabla\lambda_j \times \nabla\lambda_k - \lambda_j\nabla\lambda_k \times \nabla\lambda_i)/5 \tag{5.d}$$

It is noted that, in Kameari's element, the degrees of freedom are the circulation of the field along edges, unlike the other elements.

Adding four nodes in the element increases the dimension of the null space of the curl operator to 13 but does not affect the dimension of its range space. The curl of its element is complete to the first order in the range of the curl operator just like the other elements.

III. Application in magnetostatics

Consider a magnetostatic problem in a bounded region Ω. The boundary of Ω is split in two: $\partial\Omega = \Gamma_b \cup \Gamma_h$ and the intersection of Γ_b and Γ_h is empty. On the boundary, the boundary conditions $\mathbf{n} \cdot \mathbf{b} = 0$ on Γ_b and $\mathbf{n} \times \mathbf{h} = 0$ on Γ_h hold.

Working with the magnetic vector potential $\mathbf{a}$, the variational formulation is derived by solving weakly Ampere's theorem:

Find $\mathbf{a} \in W_2{}^1{}_b$ such that

$$\int_{\Omega} \frac{1}{\mu} \text{curl}\mathbf{a}' \cdot \text{curl}\mathbf{a} \, d\Omega = \int_{\Omega_j} \mathbf{a}' \cdot \mathbf{j} d\Omega \;\; \forall \mathbf{a}' \in W_2{}^1{}_b \tag{6}$$

where Ω_j denotes the excitation coil contained in $\Omega \cdot W_2{}^1{}_b$ is the second order edge element space including the boundary condition on Γ_b:

$$W_2{}^1{}_b = \{\mathbf{a} \in W_2{}^1 | \mathbf{n} \times \mathbf{a} = 0 \text{ on } \Gamma_b\}$$

In equation (6), the system matrix is singular because $W_2{}^1$ includes the null space of the curl operator. The solution of $\mathbf{a}$ is not unique and a gauge condition must be applied to ensure its uniqueness. Assuming Ω is topologically trivial, the kernel of the curl operator is a gradient field. The number of zero eigenvalues of the curl-curl matrix is equal to the number of nodes (including those defined on the edges and eventually on the facets) minus one. This corresponds to the dimension of null space of the curl operator. The unknowns (as well as related equations) to be removed can be thus determined by a spanning tree technique, like the case of first order element (Albanese and Rubinacci, 1990). However, the construction of a tree in the case of second order element can be very complicated. Moreover, according to the experience with first order elements, the use of a tree can cause some instability of the system and affect the accuracy of the solution. The tree technique does not seem to be the best solution.

It has been shown that convergence can be achieved without an explicit gauge condition provided that the system equation is compatible (Ren, 1996), i.e. the right hand side belongs to the range of the curl-curl matrix, or the discrete form of the RHS must be divergence free. In order to enforce compatibility, we express $\mathbf{j}$ by curl $\mathbf{t}$, where $\mathbf{t}$ is a vector potential defined in a domain Ω_t containing the coil Ω_j. It can be seen as a source field. Replacing $\mathbf{j}$ of (6) by curl $\mathbf{t}$ and integrating by parts, we get the following formulation

Find $\mathbf{a} \in W_2{}^1{}_b$ such that

$$\int_{\Omega} \frac{1}{\mu} \text{curl}\mathbf{a}' \cdot \text{curl}\mathbf{a}\, d\Omega = \int_{\Omega_j} \text{curl}\mathbf{a}' \cdot \mathbf{t} d\Omega \quad \forall \mathbf{a}' \in W_2{}^1{}_b \qquad (7)$$

This formulation is unconditionally compatible whatever the discretisation of $\mathbf{t}$. Usually, according to its nature, the vector $\mathbf{t}$ is interpolated by the edge element (first order in our application). To solve the formulation (7), no explicit gauge condition is needed when using an iterative solver (Ren, 1996).

The previously described elements have been applied to approximate $W_2{}^1$. In order to avoid the ambiguity that may occur during the assignment of degrees of freedom on the facets (in the asymmetric case of two unknowns per facet), the two basis functions w_{ijk} on a facet ijk are chosen in such a way that $i < j$ and $i < k$. This ensures a unique choice of degrees of freedom on the facets.

The following section compares the performance of these elements through an example.

IV. Comparison of results

The example to be considered is a linear magnetostatic problem. It concerns a magnetic circuit. One-quarter of the domain is shown in Figure 1. The domain is meshed by 1,050 tetrahedral elements. The mesh contains 5,824 unknowns of which 1,960 are related to edges and 3,864 to facets. In the case of Kameari's element, the number of unknowns is 7,750 of which 5,790 are related to facets. The system of equations corresponding to (7) is solved by the diagonal preconditioned conjugate gradient solver.

The results of the field (magnetic flux density) distribution given by the four kinds of elements are almost identical. This is not surprising because all these elements model correctly the range space of the curl operator with complete first order polynomials. The dimensions of the range space of the curl operator are identical for all these elements. Nevertheless, the convergence behaves very differently as can be seen in Figure 2, where the error represents the residue of the vector potential. The best convergence is obtained for Lee's element (Lee *et al.*, 1991) (which belongs to the Webb's hierarchical element (Webb and Forghani, 1993)) (curve L). The convergence of Yioultsis's element (Yioultsis and Tsiboukis, 1996) is relatively slow (curve Y). Ahagon's element (Ahagon and Kashimoto, 1995) (curve A) and Kameari's element (Kameari, 1998) (curve K)

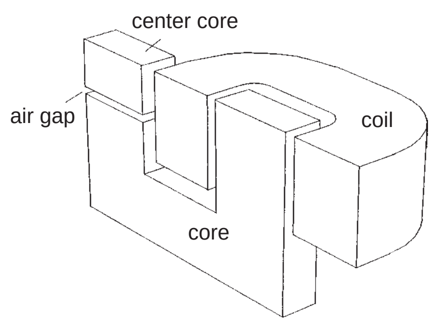

Figure 1.
Example of a magnetic circuit

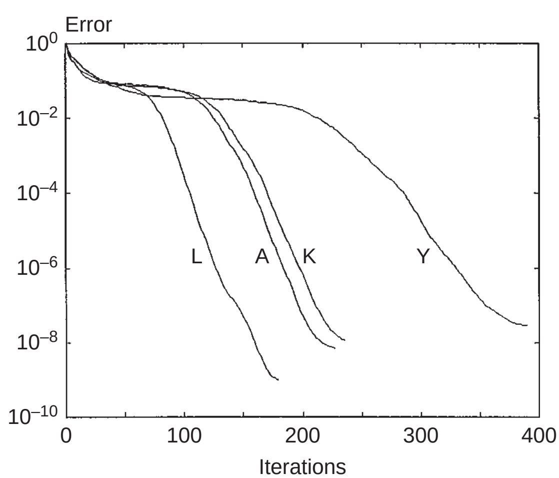

Figure 2.
Convergence behaviors of various elements

have the same order of convergence, faster than Y's and slower than L's, but K's element requires more cpu time because its number of unknowns is much higher than other elements.

In order to understand the difference of the convergence behaviors, the eigenvalues of the elementary curl-curl matrix constructed over a standard element {(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1)} are computed. There are 9 zero eigenvalues for L's, A's and Y's elements and 13 zero eigenvalues for Kameari's element. This corresponds to the null space of the curl operator as expected. The number of non-zero eigenvalues of K's element is the same as the others. This confirms that the dimensions of the range space of all these elements are the same. The non-zero eigenvalues are given in the Table I.

It can be observed that the convergence behavior of different elements is related to their maximal eigenvalue of the curl-curl matrix. This observation is different from the classical conclusion where the convergence is related to the condition number (the maximal eigenvalue over the minimal eigenvalue). It must be noted that this conclusion is true for positive definite systems. In our case, the system equation is semi-positive definite. The minimal eigenvalue is zero and the condition number is infinite. It seems that in such a case, the conditioning of matrix system and hence the convergence behavior is affected by the maximal eigenvalue. The smaller the maximal eigenvalue is, the faster the system converges. The example shows that Lee's element behaves better than other elements.

V. Comparison with first order element

A question often asked is whether the *p*-refinement technique (increasing the order of basis functions) or *h*-refinement technique (diminishing the size of elements) must be used to improve accuracy of results. In this section we try to give a comparison of the performance of second order and first order edge elements for the computation of magnetostatic fields.

In order to have a rational comparison, the refinement of the first order element mesh is arranged so that the number of unknowns is of the same order as for the second order element. The example of Figure 1 is considered. The

Lee	Ahagon	Yioultsis	Kameari
0.0083	4.8000	0.8186	10.3680
0.0351	0.6667	19.2000	10.3680
0.0023	1.2000	1.8008	10.3680
0.0049	0.8996	1.8008	0.8407
0.0172	1.1034	35.1814	1.6725
0.0204	2.6908	6.5088	1.6725
0.0319	2.8084	8.8557	4.0976
0.0380	3.9456	24.6535	4.0976
0.3566	4.0020	26.6221	12.3326
1.3652	5.9308	39.0369	31.3712
1.3868	7.1529	53.9214	31.3712

Table I.
Non-zero eigenvalues of the elemental curl-curl matrix over a standard element

first order element mesh contains 5,520 elements and 5,700 unknowns (compared with 5,824 unknowns for second order element). The convergence behavior of the first order element is compared with that of Lee's element in Figure 3.

The result shows that for the same order of number of unknowns, the two kinds of elements offer almost the same convergence behavior. But the first order element consumes much less cpu time because its stiffness matrix is more sparse than that of the second order element. In this example, the number of non-zero elements (diagonal plus the symmetric part) in the stiffness matrix is 43,828 for the first order element whereas this number is 106,888 for the second order element, more than twice of the first one. So, for the same order of number of elements, the second order element requires more cpu time and more memory space.

As concerns accuracy, the first order element provides a piecewise constant flux density field whereas the second order element gives a piecewise linear approximation. The solution behaves much better for the second order element, especially where the variation of the field is significant. This statement is clearly shown by the distribution of the magnetic flux density on a line in the air gap of the magnetic circuit (Figure 4).

To get a good solution with less cpu time and memory, the best solution is undoubtedly to mix the first and second order elements. The high order elements are to be used only where it is necessary. In this point of view, the hierarchical elements (Webb and Ida, 1993) to which Lee's element belongs, may be useful.

VI. Conclusions

Four kinds of second order edge elements have been applied to calculate magnetostatic fields. The compatibility of the formulation is ensured by introducing a source field to represent the current density and by projecting

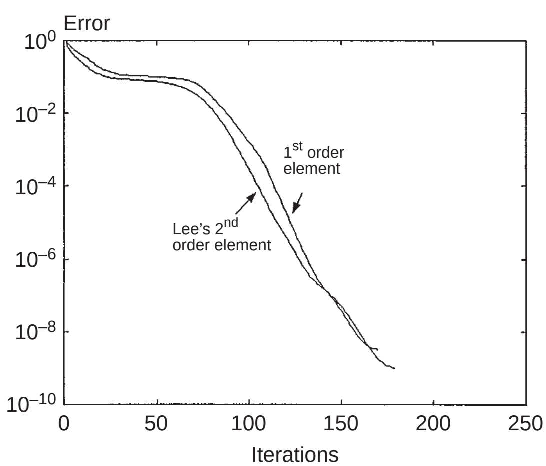

Figure 3.
Convergence behaviors of first and second order elements

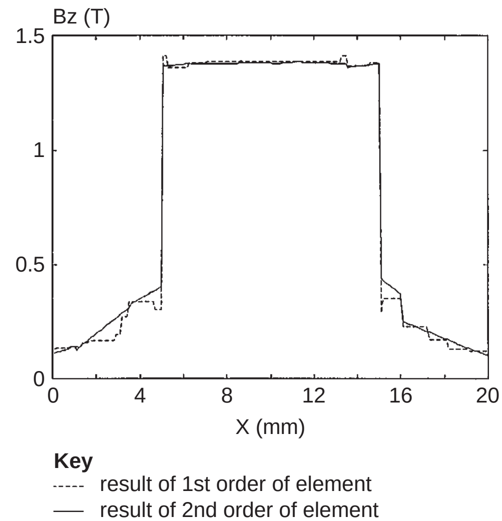

Figure 4.
Comparison of bz along a line in the airgap

this field on the curl of the element space. The convergence of the system is achieved without explicit gauge condition.

Through a comparison on a magnetic circuit problem using the same mesh, we conclude that all these elements provide the same accuracy as concerns the curl field (the flux density). However, the conditionings of the matrix system of these elements are very different. This is clearly illustrated by their convergence behaviors and also the difference of eigenvalues of a standard element. In this sense, Lee's element seems to be better than the others. Moreover, it is simple in form and belongs to hierarchical elements which may be helpful for mixed *h*- and *p*-adaptive mesh refinement.

A comparison with first order elements has also been carried out. Results show that for the same order of unknowns, the first order element is less time consuming because its stiffness matrix is more sparse. But second order elements provide smoother field results. This comparison confirms the necessity of mixing different orders of elements in adaptive mesh generation.

It must be indicated that the comparison given in this paper concerns the magnetostatic field case. The conclusion may change for magnetodynamic field computation. In fact, even though all these elements provide the same curl field, the approximation of the primal field by these elements may be different. Their performance for the computation of magnetodynamic fields will be the subject of further study.

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